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Vectors, change of basis and matrix representation: onto-semiotic approach in the analysis of creating meaning

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In a previous study, the onto-semiotic approach was employed to analyse the mathematical notion of different coordinate systems, as well as some situations and university students' actions related to these coordinate systems in the context of multivariate calculus. This study approaches different coordinate systems through the process of change of basis, as developed in the context of linear algebra, as well as the similarity relationship between the matrices that represent the same linear transformation with respect to different bases.

Keywords: linear algebra; similar matrices; change of basis; mathematical language; semiotic systems and registers; onto-semiotic approach

1. Different coordinate systems and change of bases

The issue of transiting between different coordinate systems, as well as the notion of dimension in its algebraic and geometric representations, are significant within undergraduate mathematics. The mathematical notion of *different coordinate systems* is introduced formally at a pre-calculus level, with the polar system as the first topological and algebraic example. The polar system is usually revisited as part of the calculus sequence; in single variable calculus, the formula for integration in the polar context is introduced, as a means to calculate area. In multivariate calculus, work with polar, cylindrical and spherical coordinates, as well as transformations in general, is taught in the context of multivariable functions. However, it is in the linear algebra context that the change of reference systems is taken to another level of sophistication when presented in terms of bases, and associated with the notions of linear independence and spanning sets. The idea that any of the infinite vectors in a finite-dimensional vector space can be represented as a linear combination of its basis elements, and that different matrices can represent the same linear transformation with respect to different bases, are very abstract mathematical topics. In addition to the aforementioned, the vectors that must be calculated are the coordinate vectors, that is, the weights of the linear combinations. The usual beginning approach is to work with linear transformations whose domain and codomain are in $\mathbf{R}^2$, and exemplify the above mentioned topics geometrically.

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It was in this context that the questions on the interview instrument were formulated, even though the students had finished a second linear algebra course, and had been exposed to advanced linear algebra notions, having worked in abstract as well as higher dimensional Euclidean vector spaces.

2. Linear algebra learning

The history of research related to the teaching of linear algebra in the US is usually associated with the Linear Algebra Curriculum Study Group which has been active since the 1990s [1]. The works of Carlson [2] and Carlson et al. [3] are a standard reference for those interested in different aspects that traditionally cause problems for students. The consensus is that the operational aspects of linear algebra are not problematic, but the abstraction required to understand such notions as vector space, basis and linear dependence is often overwhelming for the student. On the other hand, the group represented by Dorier [4] has carried out research projects on the teaching and learning of linear algebra since the late 1980s. One of their conclusions is that the difficulties that students have with the formal aspects of linear algebra are content-specific. In other words, while in the US it is thought that part of the problem with the linear algebra courses is that they are introduced to students who have had very little experience with abstract mathematics and proofs [5], Dorier and co-workers [6,7] affirm that in France, where the students taking linear algebra have had more exposure to formal methods in mathematics classes, the same problems occur. The conclusion that the difficulties are content-specific comes about after carrying out a historical analysis in which it is clear that '... the basic idea of linear dependence was not so easy to formalize, even by great mathematicians like Euler' (p. 187). For the purposes of this article, it is interesting to quote from Dorier and co-workers that their '... teaching experiment pays much attention to changes in mathematical frameworks, semiotic registers of representation, languages or ways of thinking.' (p. 105). In particular Alves-Dias [8] reports that 'Among the difficulties identified in linear algebra are: the number of new words to learn, comparable to a foreign language, (and) the totally new methods of exposition and demonstration...' (p. 256). This author also analysed questionnaires applied to 46 students, 23 of whom had taken a complementary-training course. Some of the questions consisted of giving short explanations in which the 'analyses bring out the misuse of terminology: students do not hesitate, for example, to speak of the kernel of a family of vectors or of a subspace'.

Parraguez and Oktac [9] carried out a study in which they gave students a set and a binary operation. Students were asked to define the 'second' operation in such a way that the set, with the two operations, would satisfy the axioms of a vector space. They concluded, among other things, that '... there should be special emphasis put on the construction of the binary operation schema, giving students the opportunity to experiment with different kinds of sets and binary operations' (pp. 2123–2124). Their study emphasized the cognitive difficulties of this task.

Oktac [10] traced the development of student interaction during an online course, in which they analysed a problem related to the eigenvalues of two matrices, 'AB' and 'BA'. She also emphasized that 'One of the issues that has been identified as problematic in introductory courses is student difficulties with logic, but as we shall see, the nature of these difficulties could be quite different from what we as the instructors assume.' (p. 444).

Sierpinska [11], in reference to linear transformations, mentions that 'Thinking in terms of prototypical examples, rather than definitions, became an obstacle to our students' understanding' (p. 222). A rather lengthy quote from Hillel [12] summarizes the role that epistemological obstacles [13] seem to play in the difficult process of generalization. The reference to the notion of vector and the learning of the general theory is of special relevance to this article.

> Thus, in a typical linear algebra course, we see two types of epistemological obstacles. The first stems from students' familiarity with analytic geometry and standard coordinates. Thinking about vectors and transformations in a geometric context certainly links these notions to more familiar ones. However, such geometric level of thinking . . . can become an obstacle to thinking about basis (rather than axes) and about the need for changing basis. The other obstacle comes about because specific notions related to $\mathbf{R}^n$ are learned by students. These notions do solve a variety of problems which are . . . linked to the central notion of systems of linear equations. Hence, this algebraic level of description becomes an obstacle to learning the general theory and to the acceptance of other kinds of objects such as functions, matrices, or polynomials as vectors. [12, pp. 205–206]

Hillel [12] is the closest antecedent we know to this study, in terms of content. Apart from analysing the different modes of description, the language and the problem of representations in linear algebra in general, he specifically deals with the problem of the representation of a linear transformation in terms of basis, in Euclidean space (pp. 201–205). Both studies have identified certain issues of the learning of this particular linear algebra aspect. However, the frameworks for analysis are not the same, and the actual analysis of that study was based on class activities and exam questions, not on a specifically designed instrument. The framework for analysis, on which this study is based, is developed in the following section.

3. Conceptual framework

The goal of this study is to analyse the behaviour of a group of students in relation to certain Linear Algebra topics (vectors, change of basis, matrix representation of linear transformations), using the theoretical and methodological tools of the 'Onto-semiotic Approach' (OSA) [14], which has been theoretically established and empirically tested. In this section, we offer a brief description of the framework, according to the following design[1]:

(1) The 'object' notion (*what is a mathematical object?*) and definition of a mathematical ontology (*which primary objects should be considered?*)

In OSA, a mathematical object is anything that can be used, suggested or pointed to when doing, communicating or learning mathematics. OSA [15,16] considers six primary objects: *language* (terms, expressions, notations and graphics); *situations* (problems, extra or intra-mathematical applications, exercises, etc.); *definitions* or descriptions of mathematical notions (number, point, straight line, mean, function, etc.); *propositions*, properties or attributes, which usually are given as statements; *procedures* or subjects' actions when solving mathematical tasks (operations, algorithms, techniques and procedures) *and arguments* used to validate and explain the propositions or to contrast (justify or refute) subjects' actions.

> Other aspects, as important as the mathematical objects, are as follows: 1) the agents that move them and the meaning (straightforward or not) that is assigned to them; 2) the concrete appearance of these objects and the reference to ideal entities; and 3) their contextual and relational function with other mathematical objects. [17, pp. 141–142, 18]

(2) The ecology of mathematical objects (*What is the contextual and functional reality of the objects?*)

In OSA, the mathematical objects are analysed by the five following dual dimensions [15, p. 5]: personal/institutional; ostensive/non-ostensive; example/type; unitary/systemic; expression/content.

> These dual dimensions demonstrate how the primary [objects] must not be understood in an isolated manner, but according to their function and their relation in a contextualized mathematical activity. Furthermore, the primary [objects] and the dualities offer a 'photographical' way of seeing the didactical systems, that is, they permit the elaboration of models that capture a changing and dynamic reality. In fact, they are indicators for the identification of the basic processes of mathematical activity. [17, p. 142]

(3) Relation between the objects (what processes are established and how do the mathematical objects relate to each other?)

Given that the objects are emerging from the system of practices, and that this emergence takes place as time goes by, the distinction *object-process* can be introduced in a natural way. Every type and subtype of a mathematical object is associated with its corresponding process (problematization, definition, argumentation, particularization, generalization, etc.). We summarize in Figure 1 the mathematical objects and processes.

The analysis and solutions detonate the use of different notions, procedures, propositions and previous arguments, and open the possibility that new ones emerge. The activation of these emerging objects is brought about by the processes of definition, of creation of techniques (algorithmic or not), the determination of propositions and argumentation. All these processes are only possible through the use of language in different registers, that is, the use of languages that make the codification and transference of knowledge and meanings of the mathematical objects involved possible. For this reason, the problem-situation is placed in the centre of the onto-semiotic analysis.

The initial objective of OSA was to give a reasonable answer, for mathematics education, to the question: what is the meaning of a mathematical notion? Godino and Batanero [19] proposed a pragmatic-anthropological answer: the meaning of any mathematical object is the system of practices (operative and discursive) that a subject carries out to solve a certain type of problem in which the object is present. These types of correspondences, dependence relationships or functions, between an antecedent (expression, representative and significant) and a consequent (content and significance), are established by some subject (person or institution) according to certain criteria or correspondence codes, and are named *semiotic functions*.

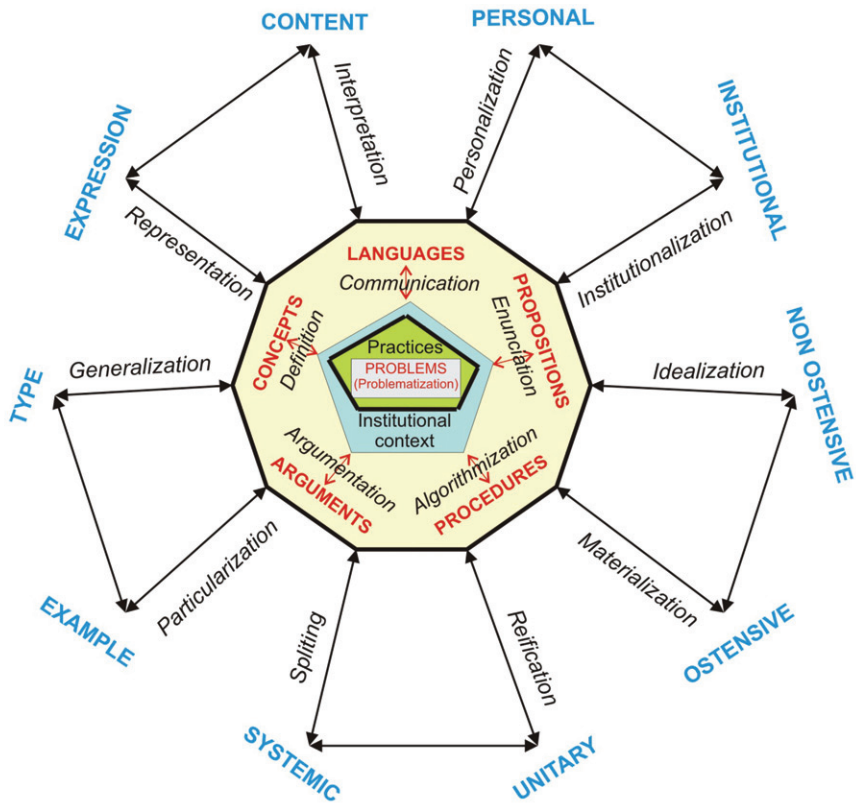


Figure 1. Objects and processes.

These semiotic functions connect the objects amongst themselves and to the practices from which they originate.

By these means, the semiotic functions, and the associated mathematical ontology, take into account the essentially relational nature of mathematics and generalize, in a radical way, the notion of representation.

The role of representation does not belong, exclusively, to language: consonant with Pierce's semiotics, it is postulated that the different types of objects (situation-problems, concepts, propositions, procedures and arguments) can be expressed or contained in the semiotic functions [20].

(4) Configurations of objects (how are the objects and processes structured during mathematical activity?)

As they relate to one another, the objects and processes form *configurations*, defined as networks of objects that intervene and emerge from the systems of practices and the relations established between them when solving a mathematical problem. These *onto-semiotic configurations* can be *epistemic* (networks of institutional objects) or *cognitive* (networks of personal objects). The *epistemic* configurations have an essential social component, given that they consist of the structuring of mathematical objects and processes (*epistemic*) recognized and shared by the agents of an institution (*socio*). For this reason, and to emphasize the social aspects in occasions, we speak of 'socio-epistemic' configurations.

Formally, a configuration can be defined by a $(n+m+r)$-tuple as

$$\begin{aligned}&(O_1, \ldots, O_n, P_1, \ldots, P_m, Sf_1, \ldots, Sf_r);\ \text{such that}\\&O_i \text{ are objects } (i \in \{1, \ldots, n\}),\\&P_j \text{ are processes } (j \in \{1, \ldots, m\}) \text{ and}\\&Sf_k \text{ are semiotic functions } (k \in \{1, \ldots, r\}).\end{aligned}$$

The '$(n+m+r)$-tuple' is a formal notion. The mathematical objects have an institutional reality that can be linked to the actual genesis of the notion, as well as to the factors that play a role in transmitting it. To make this precise, the institutional reality of the mathematical objects is linked to the:

- *Topogenesis*, that is, the 'place' where knowledge is generated. There are, essentially, three 'places': the *students* (in a learning process with an essential constructivist component), the *teacher* (in a learning process which is fundamentally magisterial) or *student–teacher* (in study processes developed according to a dynamic dialogue).
- *Cronogenesis*, that is, when the conditions for the emergence of knowledge are created, in such a way that it can be shared and dealt with on a social level.

The conditions for the emergence of knowledge in the mathematics classroom are subject to the knowledge previously shared by the students. Indeed, the same situation, introduced in two different groups, can lead to success or failure, depending on the previous study process of each one of the groups, as well as their institutional knowledge. In short, not only the knowledge to be taught, but the necessary conditions for its emergence, must be taken into account.

Tables 2, 6 and 7, that will be presented in the analysis, are formed by the objects and their temporal institutional reality (previous-emerging). The necessary processes and semiotic functions for the determination of the configurations are carried out in the socio-epistemic context (first) and then in the cognitive context (referring to the observed behaviour). The complexity forces this to occur over a period of time and not 'all at once'.

4. Research questions

(1) What primary objects and semiotic functions can be identified and classified as students confront emerging mathematical objects and must generalize previous concepts, such as vector, function, composition of functions to abstract vector, linear transformation and multiplication of matrices?
(2) Does the language of commutative diagrams enhance the understanding of the change of basis and compositions in general?
(3) How does the unitary/systemic duality influence performance and understanding of the similarity between two matrices that represent the same linear transformation with respect to different bases?

5. Context, methodology and instrument

The context of this study is a second linear algebra course, which is part of the core requirement for mathematics majors at a large public research university in the southern US. The course is cross-listed, and can serve as an upper level undergraduate course as well as a graduate course. The composition of graduate students varies from semester to semester. It is considered a required course for anyone planning to do graduate work in Mathematics and who has not had the equivalent in their undergraduate studies. Graduate students also frequently come from Masters and PhD programmes in Physics, Finance and Economics, as well as Mathematics Education.

The study relied on the following data. Four interview groups with a total of 10 undergraduates, of whom nine were Mathematics majors and one was Physics major. In these interview sessions, the students were first given a questionnaire to answer individually in a half-hour time period, after which the interviews were conducted. Written questionnaire work, without interviews, was also analysed for 14 other students, eight of whom were students of an online version of the same linear algebra course, corresponding to a Masters degree programme in Mathematics Education, and the other six were students in the same course as one of the interview groups. The interviews took place during two different semesters; for logistical reasons, there was one interview with a group of five students, two interviews with groups of two students and one interview with one student. The interviews were video-recorded and lasted approximately 2 h for the group of five students, and an hour and a half for the others. The total number of student participation, whether interview and questionnaire, or just questionnaire, was 24.

Two of the authors of this article were present as interviewers. They used a common protocol to ask specific 'probing' questions. As final grades for the course had still not been submitted, another of the authors, who was the professor of the course, did not participate in the interviews, so that the students would not feel under any pressure in terms of their grades. The students were assured that their professor would not have access to the video-recordings and their written work until after the final grades had been submitted. The book used in the course is Leon [21,22]. Table 1 contains the questions presented to the students.

In the following section, based on previous and emergent configurations, we analyse the potential answers and the observed behaviour. The first three 'pre-questions' were designed to detect the students' open-ended responses to some basic objects in linear algebra. These objects had been defined and worked with from the first course. Questions 1 and 2, which deal with change of basis, were also covered in the first course, and then revisited in the second. It is only Question 3 that presents material introduced for the first time in the second course. It covers similar matrices representing the same linear transformation with respect to different bases.

6. Analysis using the OSA

Each question will be accompanied by a table that represents the socio-epistemic[2] configuration relevant to its content and context; Questions 1 and 2 are represented in the same table. The students who were interviewed will be referred to as S1, S2 through S10, and the interviewers as I1 and I2. In the lecture sessions, the professor (one of the authors) had a well-defined system of objects and meanings that were

Table 1. Questionnaire.

A. What is a vector?
B. What is a basis?
C. Why is it important to define a basis in a vector space?
(1) Let $E = \{(1,0),(0,1)\}$, $U = \{(1,1),(-1,1)\}$, $V = \{(3,2),(4,3)\}$. If we write the vector (3,1), we can think of it as the coordinate vector with respect to the standard basis E. That is $(3,1) = 3(1,0) + 1(0,1)$.
(a) Sketch this vector with respect to the standard basis.
(b) What would the coordinate vector of (3, 1) be with respect to the basis U?
(c) Represent the new coordinate system U graphically (you can use the standard axes as a reference).
(d) What would the coordinate vector of (3, 1) be with respect to the basis V?
(e) Draw an analogous graph to (d), but with respect to the new coordinate system (you can use the standard axes as a reference).
(2)
(a) Use the following commutative triangle to help you find the change of basis matrix from U to V. Explain what you are doing:

$$\begin{tikzcd}
\mathcal{U} \arrow[rr, "U"] \arrow[dr, "V^{-1}U"'] & & \mathcal{E} \arrow[dl, "V^{-1}"] \\
& \mathcal{V} &
\end{tikzcd}$$

(b) Use the matrix to check your answers in question (1). That is, apply the change of basis matrix to take you from the coordinate vector with respect to U to the coordinate vector with respect to V.
(3) $L(x, y) = (2x - y, 3x)$ is a linear transformation from $\mathbf{R}^2 \to \mathbf{R}^2$.
(a) Find the matrix that represents the linear transformation with respect to the basis U.
(b) Once you have this matrix, how can you find the matrix that represents the linear transformation with respect to the basis V?
(c) What is the matrix that represents the linear transformation with respect to the standard basis?
(d) Describe what makes these matrices similar, and what differentiates them.

meant to be developed within the context of the institutional mathematical practices. As for the written work, reference will be made to certain tendencies. It should be mentioned that the students in the online course had access to all the lectures, as they were video-recorded; they also attended online sessions and had access to flash presentations designed by the same author, so the system was consistent in this group as well. As in Font and Contreras [20], 'the students' cognitive configuration shows some agreement with the socio-epistemic configuration, but also some differences' (p. 167). The questions can be referred to in Table 1.

6.1. *Pre-questions: previous and emergent configurations*

In this section we refer to the 'socio-epistemic' configuration, and the observed behaviours will be treated later. However, the emergence of objects is also possible here. If we understand this well, as will be made explicit further on, the emergence is not referent to new knowledge 'that emerges' from a mental process 'in real time'. In this case, a previous epistemic configuration leads to another (emerging).

We show in Table 2 some of the previous and emergent concepts, procedures, propositions and arguments to the 'pre-questions' A, B and C.

In Table 2, the emergence of the objects should be understood as the transition from one configuration to another in the configuration of semiotic functions. For example, when it is said that the notion of vector, as an ordered set of mathematical objects, emerges from the geometric notion of a vector in the Euclidean plane (or space), we refer to the configuration of semiotic functions shown in Table 3, the semiotic function B emerges from the semiotic function A.

6.1.1. *The emergence of the semiotic function 'B' from the semiotic function 'A'*

The emergence of the semiotic function B (the abstract notion of vector) has an 'institutional' and 'macro' nature. The institutional aspect follows from the fact that this emergence is made concrete in the university and not elsewhere. It is in this institution that the operational and discursive systems of practices allow its emergence or, in other words, the conditions of emergence of the different mathematical contents.

The 'macro' aspect means that the emergence of the semiotic function B is not immediate and requires a series of previous steps. The systems of practices designed in the institution will have to allow the agents in the institution to bring the evolution of the previous contents to the emerging ones. In these steps, the notions of basis and change of basis are the keys. The semiotic function B emerges after the notion of basis, as the generalization of the orthonormal basis of canonical vectors $\vec{i} = (1, 0)$ and $\vec{j} = (0, 1)$, in the conventional Cartesian plane (Table 4).

The notion of change of basis (and the matrix associated with the change of basis): coming from the expression of a vector as a linear combination of 'ordered n-tuples' that form a basis (the minimum spanning set and maximum linearly independent set) this vector is represented as a linear combination of another basis, establishing the relation between the two bases (Table 5).

This way, the configuration of semiotic functions justifies the emergence of the abstract notion of vector (semiotic function B). However, apart from this fundamental fact, it must be emphasized that the semiotic functions are 'nested' within each other to give new ones, making possible fundamental mathematical processes (generalization, in this case). Hence, the epistemic analysis of the emergence of semiotic functions can allow the analysis of certain behaviours particular to a mathematical activity. The onto-semiotic complexity that is observed here forms a reference framework that explains the observed behaviour.

However the order, or how we express it, is not the fundamental aspect. More important is the consciousness that the semiotic functions are 'nested' within each other to give new semiotic functions, permitting the key mathematical processes (generalization in this case) and making possible a more meaningful and deeper analysis of the observed behaviour.

6.2. ***Pre-questions: observed behaviour***

None of the 10 students gave an institutionally 'acceptable' definition of vector, using properties. The 10 students interviewed went from average to strong as

Table 2. Socio-epistemic configuration: previous and emergent notions, procedures, propositions and arguments to the 'pre-questions' A, B and C.

Objects	Previous	Emergent
Concepts	• Vectors as geometric objects in $\mathbf{R}^2$ and $\mathbf{R}^3$ • Vectors as 'physical' representations with magnitude and direction • The standard axes as reference in Euclidean space • Operations with vectors in $\mathbf{R}^2$ and $\mathbf{R}^3$: sum, scalar multiplication • Change of coordinate systems (rectangular, polar, cylindrical, spherical)	• Abstract vectors such as polynomials, continuous functions, matrices • Vectors as abstract objects that 'belong' to a vector space, in set-theoretic terms • The definition of basis as the smallest spanning set and largest linearly independent set • The standard basis in Euclidean space is just one of infinitely many possibilities, and not always the most adequate one • The operations of sum and scalar multiplication permit any vector in a vector space to be expressed as a linear combination of the basis elements. The weights form the components of the coordinate vectors • Change of basis
Procedures	• Operating with vectors in $\mathbf{R}^2$ and $\mathbf{R}^3$. Parallelogram law, dilatations • Finding equivalent coordinates in the different coordinate systems	• Operating with vectors in $\mathbf{R}^n$ or abstract vectors (polynomials, functions, matrices) • Expressing vectors as linear combinations of the basis elements • Finding the coordinate vectors with respect to different bases
Propositions	• A vector is an object in $\mathbf{R}^2$ or $\mathbf{R}^3$ with direction and magnitude • Cauchy–Schwarz inequality in $\mathbf{R}^2$ and $\mathbf{R}^3$ • Change of variable formulas	• Polynomial spaces, function spaces, matrix spaces, etc. are vector spaces • Cauchy–Schwarz inequality in $\mathbf{R}^n$ or abstract vector spaces • Theorems that privilege orthonormal bases, bases of eigenvectors
Arguments	• Distance formula, calculus (directional derivative, gradient) • Law of cosines to prove Cauchy–Schwarz inequality • Jacobian	• Proving that the abstract spaces satisfy the properties • Generalizing the concepts of angle and perpendicularity through orthonormal bases and the inner product

Table 3. Configuration of semiotic functions: abstract vectors.

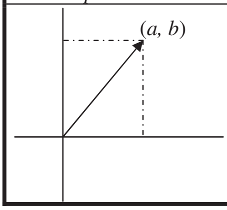
Expression B				Content B
Expression A		*Content A*		
	→	(a, b)	→	Ordered set of objects $(o_1,\ldots,o_n)$ (Abstract vectors such as polynomials, continuous functions, matrices)

Table 4. Configuration of semiotic functions: abstract vectors.

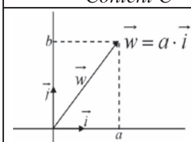
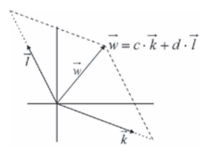
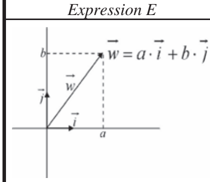
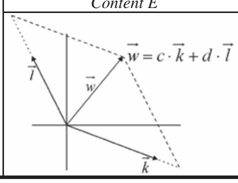
Expression D				Content D
Expression C		*Content C*		
(a, b)	→		→	

Table 5. Configuration of semiotic functions: abstract vectors.

Expression F				Content F
Expression E		*Content E*		
	→		→	$B = \{\vec{i}, \vec{j}\}$ $\downarrow A$ $B' = \{\vec{k}, \vec{l}\}$

undergraduate students, and had finished two semesters of linear algebra, in which abstract vector spaces were defined halfway through the first course. Language, definitions and propositions, as primary objects, can be used to characterize the students' semiotic functions. An analysis using the *example/type duality* sheds light on the perennial problem of transiting between the particular and the general.[3] In accordance to Figure 2, where some of the written answers are shown, emerging

S5	A vector is a direction in a space.
S1	An ordered set of numbers indicating a direction in space.
S3	A. What is a vector? A vector is description, either of a distan displacement, a change rate of change, a force, or a field, which includes information about both direction and magnitud
S2	A collection of scalars that represent a cordinate on Some Vector space as represented by some ordered basis ie. $\begin{bmatrix}3\\5\\1\end{bmatrix}_{\mathcal{E}} = 3\hat{e}_1+5\hat{e}_2+1\hat{e}_3$ $\mathcal{E}=\{\hat{e}_1,\hat{e}_2,\hat{e}_3\}$ $\begin{bmatrix}3\\5\\1\end{bmatrix}_{P} = 3x^2+5x+1$ $P=\{x^2, x, 1\}$
S4	A set of coordinates

Figure 2. Answers to the question 'What is a vector?' (evolution to the emergent semiotic function B, see Table 3).

concepts were not reflected, and the answers were phrased in terms of definitions and propositions of previous physics and calculus experience.[4] This was especially relevant, as the notion of vectors as objects in a vector space, defined in terms of properties, was emphasized and examples of the concepts introduced during the course were often given in terms of abstract vector spaces.

The answers of the group of five students that was interviewed are shown in Figure 2. These answers show the evolution of the semiotic function A to the semiotic function B (Table 3), including the misconceptions and 'relapses' to the primitive concept of vector.

In the spoken part of the interview session, the arguments to validate definitions and propositions showed different degrees of correspondence between personal and institutional (expected) meaning. For example, S2 said that a vector was:

> A collection of scalars that represent a coordinate on some vector space as represented by some ordered bases.

This was exemplified in his written answer with linear combinations of an arbitrary vector he invented, (3,5,1), with respect to the standard bases of the vector spaces $\mathbf{R}^3$ as well as $\mathbf{P}_3$, the space of polynomials of degree strictly less than 3 (we will follow the notation in Leon). S2 was the only student who made reference in the written answers to non-Euclidean vector spaces, although he gave a *characterization*, not a definition. In other words, 'the vectors characterize the polynomials' or, 'the polynomials can be characterized by vectors', but the polynomials are not conceived of as vectors. The original definition of a vector, as an arrow with direction and magnitude, still seems to override the abstract definition, given by the properties of a vector space.

On the written questionnaire, S6 defined vector as the 'direction of a line, its angle and magnitude', S7 mentioned that a vector 'has to have an origin' and S10 also affirmed that a vector 'comes out of the origin'.

When asked to elaborate in the interview, S2 said that

> It's a point that's visually hard to see. You're defining an equation by using a vector representation of an equation, but in the space of polynomials, each separate equation represents a point in that space.

It is interesting to note that S2 uses 'equation' in his oral expression as synonymous to 'polynomial'; although he is undoubtedly proficient in his elementary algebra, when using spoken language this basic knowledge is ignored.

S5, undoubtedly one of the strongest students interviewed, defines a vector, in his written answer as '... a direction in a space.' However, when asked to elaborate S5 replied:

> Yes, I was ambiguous on purpose. I know there's a lot more than the traditional explanation of a vector. It's not just one, comma one, comma, one, I know there's a lot more to it. It's a way of assigning a direction; you have to take into account the context.

If the written definition is taken as the expression, and the spoken elaboration as the content, the analysis of S5's personal meaning can be seen to correspond to what is institutionally expected, through the duality *expression/content*. The semiotic function construct is broad enough to cover the different registers (written and spoken mathematical English).

The rest of the questions do not build upon the concept of abstract vector space, and this point will not be touched upon again. However it should be mentioned that, from the first course, it was emphasized that the definition of vector was much broader than that of the vector in real Euclidean space and, as mentioned, examples and problems in the context of polynomial, function and matrix spaces were common. We definitely plan to do an in-depth study on the notion of vector space, and the obstacles that arise when generalizing the geometric version to the abstract vector space.

The pre-questions B and C asked about bases and their importance. S5 presented a peculiar written and spoken reference to linear independence (Figure 3).

The notion of equality in real numbers is given a profound and thorough treatment, using the OSA, in Wilhelmi et al. [23]. If we substitute the term 'non-equal' with the term 'linearly independent', we see that the answer is, indeed, an informal description of the notion of basis. S5's use of 'non-equal' seems to be an idiosyncratic way of referring to linear independence. However, his written answer does not conform to the institutional expectation. On the other hand, if we look at the following interchange, we can see how S5 defines a basis, correctly, when speaking:

> I1: Some of you mentioned the linear independence of vectors. Is that a condition for having a basis?

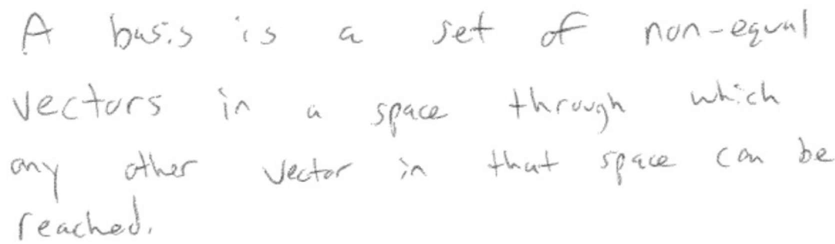

A basis is a set of non-equal vectors in a space through which any other vector in that space can be reached.

Figure 3. S5's answer to the question 'What is a basis?'.

S2: No, it's a condition for having a normal basis...
S1, S3 and S5 (simultaneously): No, it is a condition.
S3: In order to have a spanning set, the vectors need to be linearly independent...
S5: In order to have the smallest spanning set they need to be linearly independent.
S4: I think linear independence is not necessary for having a basis...
S5: The definition of a basis is the smallest...
S1: Yes, the smallest linearly independent set and the largest spanning set.
S5: No, the smallest spanning set and the largest linearly independent set...
(He then proceeded to elaborate on the reasons behind his correct spoken definition).

There are many examples which confirm the misuse of terminology quoted in Section 2. S7 stated that a basis is 'a vector that makes up the vector spaces'; however, when probed, he described a set of vectors, which spans the space and is linearly independent. S9 and S10 were interviewed together. The two talked about 'metric spaces' instead of 'vector spaces', even after two semesters of linear algebra. S10 had one of the highest grades in the course.

The pre-question C asked why it was important to define a basis in a vector space. All but S6 were clear about the spanning function of a basis. However, S1 and S2 restricted the spanning capacity to a subspace. It could be argued that they were considering the entire vector space as a subspace of itself, and that idea was not pursued. However, it is of interest to note, for future study, that somehow the notion of basis can get tied to the concept of subspace.

6.3. *Questions 1 and 2: previous and emergent configurations*

In varying degrees, the students were accurate in their calculations and sketches. The arrow notation, the labelling of the matrices and, in general, the transformational and modern approach that was emphasized in the instructional process, was applied in the written work that was analysed, although some of the students made mistakes, especially in the order of the multiplication. Concretely, in the written work, 9 out of 14 (62%) worked the problem correctly (Table 6).

On the other hand, only S4 mentioned explicitly that the basis U was orthogonal, while the basis V was not. However S4, when writing down the procedure, misused the language in such a way that, although the calculation was correct, the expression and content conveying institutional meaning was not achieved. Some explanation should be given here.

Although the change of basis is a linear transformation, the expected procedure will show the change of basis matrix applied to a coordinate vector $\vec{x}$, in this case with respect to the basis U, which will be transformed into a coordinate vector with

Table 6. Previous and emergent concepts, procedures, propositions and arguments to Questions 1 and 2.

Objects	Previous	Emergent
Concepts	• Standard basis in $\mathbf{R}^2$ and $\mathbf{R}^3$ • Change of coordinate systems (rectangular, polar, cylindrical and spherical) • Resolution of problems through equations • Composition of functions	• Change of basis to represent the same vector • Coordinate vectors • Commutative diagrams as a generalization of algebraic equations • Multiplication of the change of basis matrices representing the composition of two linear transformations
Procedures	• Calculus techniques; use of the Jacobian (determinant) as the only matrix technique • Algebraic techniques without matrices	• Systems of equations • Find change of basis matrix • Diagram chasing
Propositions	• Change of variable formulas • Equations with functions	• Those concepts related to change of basis and coordinate vectors as linear combinations of basis vectors • 'Equations' through commutative diagrams
Arguments	• Calculus argument to justify a geometric representation • Differential equations	• Linear combinations, matrix properties • Analysis of diagram chasing

respect to the basis V. The idea of representing a linear transformation with respect to two different bases is treated in the third question, together with the notion of similarity. S4, on his questionnaire, carries out the procedure of finding the change of basis matrix with success. However, S4 then tried to express the concrete result in symbolic mathematical language, which implies abstraction from the procedure. The matrix U, which represents the linear transformation 'change of basis', is replaced with a linear transformation '**L**', that 'comes from nowhere', applied to a vector $\vec{u}$ that also 'comes from nowhere'. In the context of the primary objects and the *example/type* duality, the 'situation', as an 'object, was resolved successfully with a particular numerical answer, but the 'language' object, which required abstraction, was erroneous. It is worthwhile mentioning that, without this level of abstraction, the use of technology even to solve concrete problems, in particular programmes such as *Matlab*, is severely restricted.

S3 and S5 were fluent and related situation and language through a semiotic function. The concrete calculations were the expression, and the language (mathematical English, not symbolic language as in the case of S4) was the content.

The following sequence is presented with two goals in mind for the reader. One is to illustrate the previous two paragraphs, and the other to touch on yet another mathematical notion. All interviewees, except S2, took both Linear Algebra courses with the same professor, one of the authors, and it had been emphasized since the first course that, when a matrix is associated with a linear transformation, the multiplication of matrices is analogous to the composition of functions and, in the case of vector spaces, the functions are linear transformations. However, this interpretation was not explicit in any of the students' comments.

I2: How were you supposed to use this triangle to find the change of basis?
S5: I use the little symbols.
I1: Why do we use multiplication of matrices instead of, say, addition?
S1: It's like in group theory when you say multiplication, even if the operation is addition or something else. When you see two elements separated, without a symbol, you just say multiply.
S2: The reason I say multiply is that these matrices make up a vector space of matrices and, under the axioms of a vector space, you must have addition properties and multiplication properties, like in real numbers (*here the difference between scalar multiplication, and multiplication, fell apart*).
S3: I have never given it much thought. The process reminds me of the curl.

The order of the multiplication of the matrices also came up in the interview with S9 and S10. In this case, S10 explained to S9 why U 'came before' V^{-1} in terms of 'applying' U 'first' to the vector, and then 'applying V^{-1}', but did not connect this to a composition of functions (linear transformations).

6.4. *Question 3: previous and emergent configurations*

Question 3 contains the mathematical notions that were involved in the previous questions, but its compound nature requires a layering of techniques and more complex semiotic functions (Table 7). The unitary/systemic duality, together with the reification which the systemic 'side of the coin' implies, can explain the difficulty that the students had with this question. Only S5 resolved the written problem successfully, and could communicate fluently what he had done, and why. S2 expressed the meaning of similarity correctly in mathematical English (3d), but was unable to actually solve the problem (3a, b), as he was unable to calculate the matrix representing the linear transformation with respect to the basis U. The following is an excerpt from the discussion of this question:

I1: It seems there was a feeling of uncertainty with this question.
S4: I saw a similar problem 15 minutes before the (final) exam. I realized it was a wording issue, I didn't understand what I was being asked to do.
S3: I ended up changing the vectors from the standard basis to the U basis, instead of actually doing the linear transformation.
S2: I really didn't know what was being asked. I found the representation from E (the standard basis) to U, instead of back to U. I'm in the process right now of finding from U back to U.
S1: To find the matrix representation for the linear transformation, I'm hazy on this topic. . . . I didn't know what to do. If I did have the correct matrix, to get to the linear transformation with respect to V, I would go back to my commutative triangle and apply the change of basis matrix, and that would give me the matrix representation with respect to V.
S5: To find the linear transformation with respect to the basis V you can use the same method I used on the last one, the change of basis matrix goes from V to the standard, and back to V. Or by the similarity relationship, we know how to get from U to V. . . . I put a diagram to see from what basis to what basis . . .
I1: So you actually use a little diagram.
S5: If you write the letters to let you know what you're doing, it's a lot easier to understand.

Table 7. Previous and emergent concepts, procedures, propositions and arguments to Question 3.

Objects	Previous	Emergent
Concepts	• Functions without structure • Multivariate functions with scalar codomain • Evaluation of vector functions, usually with scalar domain	• Functions with structure, linear transformations • Domain and codomain as vector spaces • Matrix representing a linear transformation • Similar matrices representing the same linear transformation
Procedures	• Evaluate functions and inverse functions • Geometric representations	• Finding change of basis matrix and its inverse • Setting up the similarity relationship to find the matrix representing the same linear transformation with respect to the other basis
Propositions	• Inverse Function Theorem	• Matrix Representation Theorem • Theorem on similarity between two matrices representing the same linear transformation
Arguments	• A function consists of a domain, a codomain and an association rule	• Existence by constructing the matrix representing the linear transformation with respect to the standard basis • The matrix is not unique, it changes with respect to the basis

All the objects, or primary objects, play some role in the analysis of this compound question. The diagrams, notation and mathematical English involved in the problem seem to privilege language together with definitions; however, the comments of the students themselves indicate that situations and procedures cannot be carried out if definitions and propositions (the emerging objects) are not contextualized in a coherent semiotic function. There is a chain of linear transformations, represented by the change of basis matrix from V to U, the matrix representing the linear transformation with respect to the basis U, and the inverse of the change of basis matrix that converts the result of the linear transformation in terms of the basis U, a coordinate vector, to one that represents the linear transformation with respect to the basis V. If each part or 'unit' is understood separately, without a systemic understanding that implies reification, the students cannot even proceed mechanically when confronted with a situation. A semester of linear algebra, plus a fourth of a second semester, is represented in this problem. When the expectations are phrased in terms of coherent semiotic functions, different levels of mastery of the mathematical objects point to the key issues in the learning of the concepts related to the change of coordinate system. These are the issues that should be addressed in the didactic process.

7. Synthesis, conclusions and prospective

Although topics in the linear algebra curriculum have been subject of important and solid studies (see Section 2), previous research, within any framework, on the mathematical concept of change of coordinate systems is practically non-existent. As was mentioned above, Hillel's [12] chapter in Dorier's [4] anthology on the teaching of Linear Algebra is the closest antecedent we have found. This study, among the other conclusions presented in this section, reinforces Hillel's observation that '... the persistence of mistakes with this kind of problem points to the existence of an obstacle that is of a more conceptual nature, and not just related to a difficulty in the operationalization of a procedure' (p. 205). In our case, this article is part of an ongoing process in which we are developing a much more sophisticated description of an epistemic configuration for this topic [17,18,24].

The transformation of expressions to content through semiotic functions, and the identification of chains of signifiers and meanings, can be accomplished because of the rich layering and complexity of these mathematical concepts. In general, the goal of the sequence of studies on change of coordinate systems that is being carried out is to create a basis for knowledge on the teaching and learning of this topic in the contexts of linear algebra and multivariate calculus.

In answer to the research questions stated in this article, we respond as follows. The primary objects that stood out when students confronted the emerging mathematical objects identified in the previous section vary according to the specific topic, although *language* is the leitmotiv that transcends in every case. It was seen that none of the students defined the notion of abstract vector using *properties* and, for that reason, did not appear to have grasped the institutional meaning. At the same time, the personal meaning of the students, when expressed in the context of the actual mathematical *situations*, showed different approximations to the socio-epistemic configurations shown in the tables. Although two participants expressed with words the fact that vectors are more than arrows in Euclidean space, the content was poor, in institutional terms, as the properties were never mentioned. However, examples abounded, in the didactical situation, where these same students had confronted situations in which the context was vector spaces of polynomials, functions and matrices.

The matrix representation of a linear transformation, in spite of the important theorem that bears its name (the matrix representation theorem), was seen as a *procedure*, in terms of primary objects. The *definitions* and *propositions* that, through semiotic functions, relate the composition of mathematical functions to matrix multiplication do not seem to have taken on the institutional meaning as emerging objects. The institutional expectation of *flexible mathematical thinking*, through which the multiplication of matrices will be understood as representing the composition of linear transformations, did not appear to be part of the students' cognitive configuration, even though it was emphasized in the didactical situation. The procedure was taught, and the steps were carried out. However, as was pointed out in the previous section, the generalization of previous objects is not only desirable, but necessary, in order to carry out institutional practices. Abstraction of concrete procedures through symbolic mathematical language is often required in the use of technology such as Matlab, even to solve concrete problems.

The commutative diagram appears to help guide students' procedures. The representational semiotic function foments systemic understanding of the process within the unitary/systemic duality, in which the arrows point to the change of bases. This was observed in all cases. Modern mathematics offers language objects that can facilitate the approximation of students' cognitive configuration to the socio-epistemic configurations for emerging objects.

Finally, it was observed that in the compound and layered context of similarity between matrices representing the same linear transformation with respect to different bases, only one of the participants resolved the unitary/systemic duality in a way that reconciled personal and institutional meaning. The fragmentation of language and *arguments* into individual units caused an impasse in the majority of students. This fragmentation, which is part of the didactic process and cannot be avoided, also conveys partial meanings. To capture the *holistic meaning* is a process that requires conscious effort on the part of the instructor, who transmits institutional meaning. The goal is to enable students to construct their personal meaning in a way that is compatible with the institutional meaning, although this is not always achieved. Only in the case of one participant were the structural semiotic functions, in which two or more objects form systems from which new objects emerge, coherent with the corresponding socio-epistemic configuration for the emerging objects.

It is essential to organize what must be known in order to do mathematics. This knowledge includes, and even privileges, mathematical concepts, and it is the search for meaning and knowledge representation that has stimulated the development of the mathematical ontology. The communication and understanding of the mathematical concepts related to the change of bases and linear transformations involve so many subsystems that it is very important not to form descriptions that are simplistic or reductionist.

The OSA gives us a framework to analyse, as mathematical objects, all that is involved in the communication of mathematical ideas as well, drawing on a wealth of instruments developed in the study of semiotics. This analysis has an essential didactical importance in relation to linear transformations: it contributes knowledge about the emergent processes of the abstract notion of vector.

This explanatory knowledge should permit the establishment of guidelines for the teaching of this notion. That is, it is *normative* or *technical* knowledge. For this reason, the need arises to design specific situations related to the 'vector' object, whose preliminary analysis can be confronted with previous knowledge (concepts, procedures, propositions and arguments) relative to the object, but whose optimal study requires emerging knowledge.

That the preliminary analysis can be confronted with previous knowledge is decisive because, implicitly, it leads to the rejection of the hypotheses of types:

- *Cognitive*. Learning is 'accumulative': knowledge is acquired once and for all, without relapses or contradictions.
- *Pedagogic*. Teaching is linear, and the correct selection, sequence and distribution in time of the knowledge are sufficient.

These hypotheses underlie a fundamental misconception: the illusion that mathematical learning can be described, by analogy, with the 'formal, axiomatic structure of the associated mathematics'. The configurations of semiotic functions that are identified determine guidelines for a *spiral curriculum* that overcomes these two false hypotheses, contradicted so often in practice.

This leads, as well, to question if an answer that adjusts exclusively to the general norm 'it is necessary to answer exactly in terms of the knowledge that the professor introduced' really can be meaningful without 'mathematical necessity'. This practice can alienate students from establishing a relation with the mathematical knowledge (that determines the mathematical necessity to act and argue in a determined way) and cause them to just proceed within the institutional restrictions imposed by the professor (the pedagogical necessity to act and argue in terms of the instructional process). Examples of this type of relation, 'without mathematical necessity' are the answers to question 1b, whose response can be given in terms pre-university elementary algebra. However, almost all the students carried out complex matrix manipulations according to the notation and methods introduced in the course. The key question is, then, the determination of situations where these notations and methods were necessary according to the mathematical point of view (efficiency, economy, generalization, etc.).

The identification and description of situations that take the configuration of semiotic functions into account, and the elaboration of a spiral curriculum that contains them (coherent with the institutional meaning pursued) is an open question that will be dealt with in future research.

Notes

1. The reader can find detailed presentations of these aspects in the cited articles, which can be found at http://www.ugr.es/~jgodino/indice_eos.htm
2. In this circumstance we emphasize the social aspect of the configuration, given that the referenced institution (Mathematics Department) is fundamental in all the analysis done.
3. The example/type duality (also called intensive/extensive) allows us to describe the different language games that arise in a particular case (e.g. the function $y = 2x + 1$) or in a more general case (e.g. the family of functions $y = \mathrm{m}x + n$). This duality allows us to explain a basic aspect of mathematical activity: the use of generic elements and the associated processes of generalization.
4. However, several of these students had performed correctly on exams where, systematically, problems related to concepts, such as linear independence, linear transformation, change of basis, or least squares, were given in the context of vector spaces of polynomials, functions, and matrices.

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