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ALGEBRAIZATION LEVELS IN PRIMARY, MIDDLE AND HIGH SCHOOL MATHEMATICS

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A synthesis of the school algebraic reasoning model developed by Godino et al. (Godino, Ake, Gonzato & Wilhelmi, 2014; Godino, Neto, Wilhelmi, Ake, Etchegaray & Lasa, 2015), which recognises six levels of algebraization at the different educational stages (K - 12 6-18 years) is presented. The distinctive features of each level are illustrated through its application in two tasks with different variants, whose resolution put at stake algebraic objects and processes specific of each level. This modelling of algebraic reasoning can help teachers identify features thereof useful for its promotion and development in the student's learning process.

ALGEBRAIC REASONING LEVELS IN PRIMARY EDUCATION

The distinction of algebraization levels applied in the paper is based on the application of the idea of intensive object introduced in the "Onto-semiotic Approach" (OSA) (Godino, Batanero, & Font, 2007; Font, Godino, & Gallardo, 2013). If an object represents a particular instance within a mathematical practice it is termed *extensive* object; while it is considered to be *intensive* when it intervenes representing a class or generality. These attributes of mathematical objects are not absolute: they are subjected to the language game in which they participate. For example, $y = 2x + 1$ is a particular function belonging to a class or type of linear functions, $y = mx + n$; the latter expression is then an intensive object. However, in the study of polynomial functions, the linear function, whose generic expression is $y = mx + n$, is a particular case (extensive) of that class of functions (intensive).

Despite the relativity of the extensive – intensive duality, it is useful to establish a graduation of generality or intension of mathematical objects, as indicated hereafter. Each particular number is already a general entity because it is abstracted from a collection of perceptible objects and actions carried out with them. We say that numbers are intensive objects with a *first degree* of generality, while perceptible objects from which they emerge have a *zero degree* of generality. The unknowns and variables (and therefore the equations and particular functions) are intensive objects with a *second degree* of generality, since they put at stake classes or types of objects of first degree of generality (usually numbers).

The attribution of an algebraic character to a mathematical practice involves the intervention of intensive objects at least with second degree of generality, in addition to the use of some kind of language and analytical calculation with these objects. Conversely, the attribution of arithmetic character to a mathematical practice involves the intervention of intensive objects with a first degree of generality (usually numbers).

Table 1 summarizes the distinctive features of the three levels of algebraization described by Godino et al. (2014), together with the level 0 (no algebraic features), which are based on the following onto-semiotics distinctions: 1) Presence of intensive objects of a second degree of generality (algebraic objects such as unknowns, variables, indeterminate); 2) Types of languages used (natural, iconic, gestural or symbolic) to denote or represent the intensive objects; 3) Treatment (syntactic or analytical calculation) applied to the objects (operations, transformations based of the application of algebraic properties of the corresponding structures).

LEVELS	OBJECTS	TRANSFORMATIONS	LANGUAGES
0	- Objects with a first degree of generality (particular numbers) -Operational meaning of equality -Variables as placeholder of particular numbers	Arithmetic operations with particular numbers	Natural, numerical, iconic, gestural
1	-Objects with a second degree of generality (sets, classes or types of numbers) -Relational meaning of equality -Variables as unknowns	Operations with objects of first degree of generality, applying properties of N algebraic structure, and equality as equivalence.	Natural, numerical, iconic, gestural; symbols can be used involving spatial, temporal and contextual information.
2	- Objects with a second degree of generality (sets, classes or types of numbers) -Relational meaning of equality -Variables as unknowns, generalized numbers and changing quantity	-Operations with objects of first degree of generality, applying properties of N algebraic structure, and equality as equivalence. -Equations are of the form, $Ax + B = C$. -In functional tasks generality is recognized but operations with variables are not carried out to get canonical forms of expressions	Symbolic - literal, used to refer to intensive objects recognized, although still linked to the spatial, temporal and contextual information.
3	Indeterminates, unknowns, equations, variables and particular functions intervene (Intensive objects with a second grade of generality).	-Operations with objects of second degree of generality, - Equations are of the form $Ax \pm B = Cx \pm D$. - Operations with indeterminates or variables are carried out to obtain canonical forms of expression.	Symbolic – literal; symbols are used analytically (meaningless), without referring to contextual information.

Table 1: Characteristics of elementary levels of algebraic thinking

Solutions to a task and theirs corresponding algebraization levels are displayed below.

Task 1. Students either go by car or they walk to a certain school. There are 3 students walking for every 3 student going by car. If the school has 212 students, how many of them use each means of transport?

Solution 1 (level 0): For every 3 students who walk, there is 1 going by car. Hence, in every group of 4 students (3 + 1) there is 1 going by car (a fourth of students). Thus, 50 out of 200 students go by car and 3 out of 12 students use the car. Therefore, 53 students use the car and three times that amount, that is, 159, walk to the school.

Solution 2 (level 1): For every 4 students there are 3 which walk. We write out the following proportion: 4 (children) ------> 3 walk ; 212 (children) ------> x walk

$\frac{4}{3}=\frac{212}{x}; x=3\times\frac{212}{4}; x=159$. Then, 159 children walk and 53 go by car.

Solution 3 (level 2): If *x* is the number of students going by car: 212=*x* + 3*x*; 212 = 4*x*;

x = 212 / 4;   x = 53. Then, 53 children go by car and 212-53 = 159 walk.

Solution 4 (level 3): Let be *x* = children going by car; *y* = children walking.

x+y = 212; *y = 3x*; *x+3x = 212; 4x = 212; x = 212/4 = 53.*

ALGEBRAIC REASONING LEVELS IN SECONDARY EDUCATION

The work with levels 1, 2 and 3 continues in secondary school; in particular, usually a central goal in the first year of this stage is achieving the mastering of level 3. The use of parameters and their treatment is a criterion to define higher levels of algebraization, as it is linked to the presence of families of equations and functions, and therefore implies new "layers" or levels of generality (Radford, 2011). The first encounter with the parameters is linked to a fourth level of algebraization and performing calculations or treatments with parameters and variables corresponds to a fifth level. The specific study of algebraic structures leads to the recognition of a sixth algebraization level of mathematical activity (Table 2). We use task 2 to exemplify these new levels.

LEVELS	OBJECTS	TRANSFORMATIONS	LANGUAGES
4	Variables, unknowns and parameters; Families of equations and functions (Intensive objects with a third grade of generality)	There are operations with variables but not with the parameters	Symbolic – literal; symbols are used analytically, without referring to contextual information.
5	Variables, unknowns and parameters; Families of equations and functions (Intensive objects with a third grade of generality)	There are operations with the parameters and hence with objects with a third grade of generality	Symbolic – literal; symbols are used analytically, without referring to contextual information.
6	Abstract algebraic structures (vector spaces, groups, rings, …) General binary relations and its properties (Intensive objects with a fourth grade of generality)	There are operations with the objects forming parts of the structures	Symbolic – literal; symbols are used analytically without referring to contextual information.

Table 2. Characteristics of algebraic thinking levels in secondary school

Task 2. You row your kayak 5 miles downstream from your campsite to a dam, and then you row back to your campsite. You row x miles per hour during the entire trip, and the river current is 1 mile per hour. Write an expression for the total time of the trip.

Expected solution, objects and levels: If $x \leq 1$, the river stream would prevent the return; therefore it is assumed that $x > 1$. It is know that the distance travelled *e* by an object in uniform motion is the speed (*x*) multiplied by time (*t*), then *t = e/x*. When rowing downstream, speed will be *x* + 1, and when it goes upstream, *x* - 1. So the time *t* to make the full tour (round-trip) depending on the speed *x* can be calculated with the expression:

$$t = \frac{5}{x+1}+\frac{5}{x-1}=\frac{5(x-1)}{(x+1)(x-1)}+\frac{5(x+1)}{(x-1)(x+1)}=\frac{10x}{(x+1)(x-1)}$$

A rational algebraic expression is found as criterion for a function whose independent and dependent variables take values in the set of positive real numbers greater than 1 ($x > 1$). Operation with the independent variable to get a canonical expression is done, and hence the level 3 of algebraization is involved.

Task variants: The rowing speed data can be given (e. g, 4 miles per hour) and ask for the time the trip takes; in this case the solutions only requires arithmetic calculations (level 0). On the other hand, the distance to the camp can be considered as variable, as well as the speed of the river flow. In this case the functional expression involves the use of two parameters (level 4). Given the time as data and asking for the river flow speed, or the distance to the camp, will make necessary to find these parameters (level 5). Finally, generalization to other contexts whose modelling requires other polynomial or rational functions can be proposed; in this case, a foundation in terms of functional algebra would be necessary (level 6).

SOME IMPLICATIONS FOR TEACHERS EDUCATION

These algebraic reasoning levels have implications for training, both primary and secondary school teachers. In addition to develop curricular proposals (NCTM, 2000) including algebra from the earliest levels of education, the teacher need to act as the main agent of change in the introduction and development of algebraic reasoning in elementary classrooms, and its progression in secondary education. Reflecting on algebraic thinking objects and processes and recognising them can help identify the features of mathematical practices on which the teachers can intervene to gradually increase the algebraization levels of students' mathematical activity

Considering algebraization levels of mathematical activity can help raise awareness of gaps or discontinuities in didactical trajectories. These gaps involve the use of different registers of semiotic representation, their treatment and conversion, as well as the establishment of relations between conceptual, propositional, procedural and argumentative objects of higher generality.

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