What Makes Mathematics Teacher Knowledge Specialized?

Offering Alternative Views[1]

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Abstract. *The purpose of this article is to contribute to the dialogue about the notion of mathematics teacher knowledge, and the question of what makes it specialized. In the first part of the article, central orientations in conceptualizing mathematics teacher knowledge are identified. In the second part of the article, alternative views are provided to each of these orientations that direct attention to underexplored issues about what makes mathematics teacher knowledge specialized. Collectively, these alternative views suggest that specialization cannot be comprehensively accounted by addressing 'what' teachers know, but rather by accounting for 'how' teachers' knowing comes into being. We conclude that it is not a kind of knowledge but a style of knowing that signifies specialization in mathematics teacher knowledge.*

Keywords: mathematical knowledge for teaching; pedagogical content knowledge; specialized knowledge; teacher knowledge; teacher professionalism

Introduction

Mathematics teacher knowledge has become a fertile research field in mathematics education (see Ponte & Chapman, 2016). Scholars have considered mathematics teacher knowledge from multiple perspectives, using various constructs and frameworks to describe and explain what makes mathematics teacher knowledge specialized[1]. Despite the relatively short time that research on teacher knowledge has existed as a field, the literature is currently shaped by a diversity of conceptualizations of mathematics teacher knowledge (Petrou & Goulding, 2011; Rowland, 2014).

As research on teacher knowledge has moved to a more central role in mathematics education research (see Ball, Lubienski, & Mewborn, 2001; Even & Ball, 2010; Fennema, & Franke, 1992; Sullivan & Wood, 2008), the search for what signifies the specialization in mathematics teacher knowledge has been becoming an increasingly important enterprise in the research field. Recent research has addressed this issue by describing and identifying facets or types of teacher knowledge that have been considered as crucial for teaching mathematics, and in obtaining empirical evidence to support these (e.g., Ball, Thames, & Phelps, 2008; Baumert et

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al., 2010; Blömeke, Hsieh, Kaiser, & Schmidt, 2014). As such, the focus tends to be on (seemingly distinct) facets of knowledge that an individual teacher possesses (knowledge for teaching) or uses in the classroom (knowledge in teaching). A number of scholars have pointed to inadequacies in such conceptualizations of teacher knowledge, arguing that they disregard the deep embeddedness of knowledge in professional activity (Hodgen, 2011) and ignore the dynamic interactions between different kinds or facets of teacher knowledge (Hashweh, 2005). Others have argued that the premises on which much research into teacher knowledge is based depend on assumptions that are rather aligned with transmission views of teaching (McEwan & Bull, 1991) and, in consequence, are rather asymmetrical to constructivist viewpoints (Cochran, DeRuiter, & King, 1993). Thus, it is not surprising that scholars have called for making the assumptions underlying frameworks of teacher knowledge, teaching, and teacher learning explicit (Lerman, 2017) and for achieving coherence between research into teacher characteristics and teacher practice (Van Zoest & Thames, 2013).

This paper aims to make explicit the discussion of what makes mathematics teacher knowledge specialized, a question that has often been addressed implicitly by several scholars in various ways and with different emphases. The paper outlines further attempts that reflect theoretically on this important issue and try to articulate more explicitly what it is, or may be, that signifies the specialization in mathematics teacher knowledge. The purpose of this paper is, therefore, twofold: First, we try to elucidate central orientations currently available in the literature and point to the more serious limitations of the grounds on which they stand. Second, we provide alternative views that direct attention to underexplored issues about these orientations.

We begin this article by briefly discussing previous accounts on what mathematics teacher knowledge signifies and encompasses, and then take this retrospection as a point of departure for outlining the limitations of these accounts. Afterwards, we articulate and draw a contrast with alternative viewpoints that provide a critical stance towards previous accounts but also provide new ways to think about the issues under consideration. The first perspective underlines the complex dynamics of the usage and function of mathematics teacher knowledge in context that calls for specialization as a process of becoming rather than a state of being. The second perspective points to the epistemological stance inherent in mathematics teacher knowledge, arguing for the sensitivity for the historical and cognitive geneses of mathematical insights. The third perspective accentuates the complex interactions of knowledge facets that generate dynamic structures. Then, we highlight underlying themes and convergences of these alternative views with regard to specialization in mathematics teacher knowledge. Finally, we conclude by proposing to construe specialization in mathematics teacher knowledge as a style of knowing rather than a kind of knowledge.

On the Evolution of Thinking within the Field Regarding Conceptualizing Mathematics Teacher Knowledge

Research into mathematics teacher knowledge has evolved considerably, especially over the last three decades. The number of studies in this field has significantly increased, the nature and scope of the research have expanded, and the frameworks used to guide the study of mathematics teacher knowledge have become quite diverse. The growing diversity of frameworks for teacher knowledge testifies to the complexity and multidimensionality of the research field.

To identify current views in the literature concerning what makes mathematics teacher knowledge specialized, we try to briefly sketch the evolution of thinking within the field in conceptualizing mathematics teacher knowledge. We acknowledge that a great deal of important detail is lost in the brief sketch of this development. More detailed accounts of this research can be found elsewhere (see e.g., Kaiser et al., 2017; Kunter et al. 2013; Rowland & Ruthven, 2011; Schoenfeld & Kilpatrick, 2008). A recent discussion of several research traditions is provided by Blömeke and Kaiser (2017), in which the same authors arrive at a complex framework of teacher competence and conceptualize the development of teacher competence as personally, situationally, and socially determined, as well as embedded in a professional context.

Our purpose here, however, is to foreground how the field in general has induced particular attitudes towards what mathematics teacher knowledge signifies. We start by portraying different dimensions of mathematical knowledge discussed in the literature as being essential for mathematics teachers. Then, we draw attention to selected contributions that articulate what particularizes subject matter knowledge for teaching, particularly in reference to mathematical knowledge for teaching, with an emphasis on the way specialization is considered. Afterwards, we focus on what is considered as the heart of teaching: the *transformation* of subject matter in ways accessible to students, an assumption that underlies several attempts in conceptualizing mathematics teacher knowledge.

Mathematical Knowledge

The literature foregrounds different aspects of mathematical knowledge as important for teachers. Shulman (1986), for instance, accredited “the amount and organization of the knowledge per se in the mind of the teacher” (p. 9), referring to Schwab’s (1978) distinction between substantive and syntactic structures of a discipline. Substantive structures are the key concepts, principles, theories, and explanatory frameworks that guide inquiry in a discipline, while syntactic structures provide the procedures and mechanisms for the acquisition of knowledge, and include the canons of evidence and proof. Bromme (1994), then again, acknowledged that “school subjects have a ‘life of their own’ with their own logic; that is, the meaning of the concepts taught cannot be explained simply by the logic of the respective scientific disciplines” (p. 74). In recognizing school mathematics as a special kind of mathematics, Bromme (ibid.) suggested school mathematical knowledge and academic content knowledge as further dimensions of mathematical knowledge. Buchholtz et al. (2013) set forth a kind of knowledge “that comprises school mathematics, but goes beyond it and relates it to the underlying advanced academic mathematics” (p. 108). The same authors called this kind of knowledge, in homage to the pioneering work of Felix Klein, knowledge of elementary mathematics from an advanced standpoint.

This small selection of a fuller corpus of dimensions of mathematical knowledge already indicates a critical point to be expanded here: the contributions to dimensions of mathematical knowledge that teachers know, or should know, is accumulative (or incremental). However, as Monk (1994) reminds us, “a good grasp of one’s subject areas is a necessary but not sufficient condition for effective teaching” (p. 142). We might interpret Monk’s statement as a call for additional knowledge, but we might also understand it as a call for a *qualitatively* different kind of knowledge.

Subject Matter Knowledge for Teaching (Pedagogical Content Knowledge)

A critical advance in the field was the recognition that teaching entails a specialized kind of subject matter that is distinct from disciplinary subject matter. Shulman (1986) proposed a kind of knowledge "which goes beyond knowledge of subject matter *per se* to the dimension of subject matter knowledge *for teaching*" (p. 9, italics in original) that he labeled *pedagogical content knowledge* (PCK). Shulman (1986) described PCK as encompassing

"for the most regularly taught topics in one's subject area, the most useful forms of [external] representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations – in a word, the ways of representing and formulating the subject that make it comprehensible to others […] [and] an understanding of what makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons." (p. 9)

In this view, PCK consists of two dimensions: 'knowledge of representations of subject matter' and 'knowledge of specific learning difficulties and students' conceptions'. These two dimensions often served as reference points in thinking about PCK, as Ball (1988), for instance, assumed "[…] 'forms of representation' […] to be the crucial substance of pedagogical content knowledge" (p. 166). She then explored the more dynamic aspects of this idea, examining pre-service teachers' pedagogical reasoning in mathematics as the process whereby they build their knowledge of mathematics teaching and learning. Other scholars in mathematics education have delineated dimensions of PCK that extended or refined Shulman's original considerations. For instance, Marks (1990) clarified PCK in the context of mathematics by identifying four dimensions, including knowledge of students' understanding, knowledge of subject matter for instructional purposes, knowledge of media for instruction, and knowledge of instructional processes.

Shulman (1987) asserted that among multiple knowledge domains for teaching (e.g., content knowledge, general pedagogical knowledge, curriculum knowledge, knowledge of learners, etc.), it is PCK that is "the category most likely to distinguish the understanding of the content specialist from that of the pedagogue" (p. 8). As such, the existence of PCK relies on and projects the belief in a distinction between the subject matter knowledge of teachers and that of other subject specialists or scholars (e.g., mathematicians). While the notion of PCK advocated a position distinguishing teachers' and academics' subject matter knowledge, the concept of *mathematical knowledge for teaching* advocated a position distinguishing knowledge for teaching mathematics from knowledge for teaching other subjects (such as physics, biology, or the arts).

Mathematical Knowledge for Teaching

The notion of *mathematical knowledge for teaching* has become an important point of departure in thinking about what signifies the specialization in mathematics teacher knowledge. Various researchers have applied different emphases to this notion, as shall be seen below. In this realm, it is particularly the *Mathematical Knowledge for Teaching* (MKT) framework (e.g., Ball & Bass, 2000; Ball et al., 2008), that has attracted significant research attention. The MKT framework evolved through the application of a kind of job analysis (Ball et al., 2008) focusing on the use of knowledge in and for the work of teaching.

The MKT framework defines several sub-domains within two of Shulman's (1987) original knowledge domains: pedagogical content knowledge (PCK) and subject matter knowledge

(SMK). PCK is divided into knowledge of content and students, knowledge of content and teaching, and knowledge of curriculum, whilst SMK is divided into common content knowledge, specialized content knowledge, and knowledge at the mathematical horizon. We briefly outline four of the six dimensions, excluding horizon content knowledge and knowledge of curriculum as they have so far not been the primary focus of studies into the area.

Within PCK, *knowledge of content and teaching* combines knowing about teaching and knowing about mathematics, including knowledge of the design of instruction, such as the knowledge governing the choice of examples to introduce a content item and those used to take students deeper into it. *Knowledge of content and students* is the knowledge that combines knowing about mathematics and knowing about students. It includes knowledge of common student conceptions and misconceptions about particular mathematical content as well as the interpretation of students' emerging and incomplete thinking.

Within the mathematical knowledge domain, *common content knowledge* refers to the mathematical knowledge and skill possessed by any well-educated adult, and certainly by all mathematicians, which is used in settings other than teaching. *Specialized content knowledge*, on the other hand, is defined as mathematical knowledge tailored to the specialized uses that come up in the work of teaching. It is described as being used by teachers in their work, but not held by well-educated adults, and is not typically needed for purposes other than teaching. Ball et al. (2008) noted that teaching may require "a *specialized* form of *pure* subject matter knowledge" (p. 396, italics added):

"*pure* because it is not mixed with knowledge of students or pedagogy and is thus distinct from the pedagogical content knowledge identified by Shulman and his colleagues and *specialized* because it is not needed or used in settings other than mathematics teaching." (Ball et al., 2008, p. 396, italics added)

Transforming Subject Matter

These approaches support the assertion that a kind of subject matter knowledge *exists* that is qualitatively different from the subject matter knowledge of disciplinary scholars or teachers of other subjects. The nature of such knowledge, however, is not just a matter of mastering disciplinary subject matter. From the perspectives presented so far, teachers' primary concern is not with mathematics, but with teaching mathematics. The difference between disciplinary scholars and educators is, therefore, also seen in the different uses of their knowledge. This important recognition of the different purposes of disciplinary scholars and teachers highlights, as Shulman (1987) argued, a unique aspect of teachers' professional work: a teacher must "transform the content knowledge he or she possesses into forms that are pedagogically powerful and yet adaptive to the variations in ability and background presented by the students" (Shulman, 1987, p. 15). It is this notion of *transforming* the subject matter of an (academic) discipline that highly impacted our thinking about teacher knowledge, but it seems to have been taken for granted once the picture of knowledge for teaching was defined. The primary purpose of transformation is to organize, structure, and represent the subject matter of an (academic) discipline in a form "that is appropriate for students and peculiar to the task of teaching" (Grossman, Wilson, & Shulman, 1989, p. 32).

The literature on mathematical knowledge for teaching also identifies various discipline-specific practices of transformation, often described in terms of exemplifying, explaining, decompressing, or simplifying, that converge on teachers' core practice of unpacking mathematics content in ways that are accessible to students (Adler & Davis, 2006; Ball & Bass, 2000; Ma, 1999). It requires the capacity "to deconstruct one's own mathematical knowledge into a less polished and final form, where elemental components are accessible and visible" (Ball & Bass, 2000, p. 98). Hodgen (2011), for instance, takes this idea further arguing that the "essence of teacher knowledge involves an *explicit* recognition of this – 'unpacking' the mathematical ideas […], [whereas] doing mathematics only requires an *implicit* recognition of this." (pp. 34-35, italics in original).

More recently, the idea of transformation has also been further elaborated by scholars working in the *Knowledge Quartet* research program (Rowland, 2009, Rowland, Huckstep, & Thwaites, 2005), as part of their conceptualization of the classification of situations in which mathematical knowledge surfaces in teaching. The research group considers transformation as concerning "knowledge in action as demonstrated both in planning to teach and in the act of teaching itself. A central focus is the representation of ideas to learners in the form of analogies, examples, explanations, and demonstrations" (Rowland, 2009, p. 237). This conceptualization concerns knowledge in action, focusing on teaching activity in the transmission of content.

Thinking About What Makes Mathematics Teacher Knowledge Specialized: Various Orientations, Different Responses

As innocent and straightforward as the question *What makes mathematics teacher knowledge specialized?* sounds, the research field has found it difficult to provide an explicit answer as there are various orientations towards teacher knowledge, each with a quite different response to the question. The previous section briefly outlined the following orientations regarding what mathematics teacher knowledge signifies: (1) identifying and describing multiple dimensions of mathematical knowledge (and pedagogical content knowledge), (2) declaring kinds of subject matter knowledge for teaching that are distinct from subject matter knowledge per se, and (3) asserting teachers' action upon subject matter (that is the transmission of subject matter in ways accessible to students) as the core task of teaching.

These three orientations seem to indicate different lines of thinking about what makes mathematics teacher knowledge specialized. Each focuses attention on particular aspects: the first considers additional knowledge dimensions (quantity), whereas the second turns the attention towards knowledge that is construed as qualitatively different. These different lines of thinking seem to be convolved in Shulman's idea of transforming subject matter, that is, the various orientations shape, and are shaped by, our interpretations of Shulman's idea of transforming subject matter.

One might interpret Shulman's (1986, 1987) initial writings on teacher knowledge as indicating a stance in which teachers' and disciplinary scholars' subject matter knowledge were differentiated, signifying the existence of a kind of subject matter knowledge for teaching (held by teachers) that is qualitatively different from subject matter knowledge per se (held by disciplinary scholars). On the other hand, Ball and her colleagues proposed a more nuanced differentiation in which subject matter content itself is considered in a way that only makes sense to *mathematics* teachers. In other words, while both notions of PCK and specialized content knowledge indicate the existence of a qualitatively different kind of knowledge, they differ in where to put emphasis: Shulman's notion of PCK puts emphasis on a kind of knowledge distinctive to *teachers* (and not to disciplinary scholars) and Ball and her colleagues' notion of specialized content knowledge puts emphasis on a kind of knowledge distinctive to *mathematics* teachers (and not to teachers of other subjects).

Each of these orientations provides a (partial) response to the question of what signifies mathematics teacher knowledge. The first orientation calls for the multidimensionality of mathematical knowledge in particular, and teacher knowledge in general. The second orientation argues for the qualitative differences between scholars' subject matter knowledge (per se) and teachers' subject matter knowledge (for teaching) or the qualitative differences between knowledge for teaching mathematics and knowledge for teaching other subjects. The third orientation, underlying and extending the previous one, points to teachers' actions upon subject matter, as manifested in notions such as transforming, unpacking, deconstructing, and decompressing subject matter.

Correspondingly, we can frame the responses of the various orientations concerning what makes mathematics teacher knowledge specialized as follows:

- mathematics teachers need to know more than the subject matter they teach (*additional knowledge*);
- mathematics teachers need to know subject matter in a qualitatively different way than other practitioners of mathematics (mathematicians, physicists, engineers, among others), and they need to hold a qualitatively different kind of knowledge than teachers of other subjects (physics teachers, biology teachers, history teachers, among others) (*qualitatively different knowledge*); and
- mathematics teachers need to know how to organize or structure the subject matter in ways accessible to students (*teaching-oriented action*).

These responses, taken together, seem to converge on an understanding that what mathematics teacher knowledge signifies depends on its *distinctiveness* or *exclusiveness*: mathematics teacher knowledge is construed as knowledge that is needed *only* for teaching mathematics, that is, knowledge not required for other purposes than teaching and not needed for teaching other subjects than mathematics.

Too often when we frame our thinking about what mathematics teacher knowledge signifies we see ourselves getting caught in the mire of current debates without taking a critical stance toward the grounds on which they stand. In the present paper, it is intended to take a more critical stance toward the current state of what the literature implicitly represents as making mathematics teacher knowledge specialized. To this end, we explicitly identify the more significant boundaries demarking the outlined orientations, and provide new ways of thinking about the issue under consideration. Our critique rests on at least three general tendencies that seem to have been implicit in the current discussion on teacher knowledge:

- the field brings up external references in justifying what makes teacher knowledge specialized (mathematics teachers vs. mathematicians; teaching mathematics vs. teaching other subjects);
- in its consideration of teacher knowledge, the field takes a disciplinary perspective which is structuralist[2] in orientation, arguing from the viewpoint of teaching mathematics; and

- the field has been partly additive, that is, accumulating dimensions of teacher knowledge.
- In the following sections, we adopt a critical stance to these general tendencies, around which we organize our understanding of the question of what makes knowledge for teaching mathematics specialized. As such we argue for an approach which is:
- intrinsic: it dispenses with external reference points, and accounts for specialization as a process of becoming rather than a state of being;
- anthropological-sociocultural: it eschews a restrictive structuralist approach, and instead underlines the epistemological thread inherent in mathematics teacher knowledge; and
- transformative: rather than seeing teacher knowledge as an incremental accumulation of facets, it accentuates the complex interactions of knowledge within a dynamic structure.

In doing so, we draw on and debate different emerging perspectives that provide critical issues that are un- or under-addressed in the current literature, and, more importantly, that provide provocative new avenues for thinking about what makes mathematics teacher knowledge specialized in ways not yet explicitly articulated.

From an Extrinsic to an Intrinsic Approach

In this section, we adopt a critical stance to a tendency that seems to be common amongst scholars discussing mathematics teacher knowledge: the tendency of comparing mathematics teacher knowledge with the knowledge demanded of other professionals (such as mathematicians, teachers of subjects other than mathematics, etc.). Such an approach is extrinsically oriented (see Flores, Escudero & Carrillo, 2013) as it takes an external referent (e.g., mathematicians or teachers of other subjects) as a reference point for comparison. The explicit purpose of such an approach is to identify the distinctiveness of mathematics teacher knowledge in relation to someone else's knowledge.

Since Shulman (1986) acknowledged teachers as professionals, various scholars in mathematics education have attempted to identify the distinctiveness of knowledge for teaching mathematics in comparison with other forms of knowledge. This search took place primarily by looking outside of mathematics education to provide answers as to what mathematics teacher knowledge signifies. Researchers articulated ways in which mathematics teacher knowledge differs from mathematicians' knowledge, or how it differs from knowledge of those who teach subjects other than mathematics. This tendency to look beyond the discipline, we believe, is a very natural one, particularly when, at the same time, scholars were searching for an identity for the research field. In relating mathematics teachers to professionals of other disciplines, scholars were able to determine certain cognitive dispositions that seemed to be specific for mathematics teachers – aspects of teacher knowledge that have been referred to as being static, explicit, and objective (in the sense of being observable). However, it is one thing to make comparisons between mathematics teacher knowledge and the knowledge pertinent to other professionals, and quite another to interpret the seemingly distinctive features of teacher knowledge in terms of 'specialization'. Whereas 'specialization' seems to have been understood in terms of distinctiveness, in this paper, we argue for a different meaning of specialization that allows us to focus our attention inside and not necessarily outside.

Flores et al. (2013), for instance, identified difficulties in defining the specialized nature of certain cognitive dispositions when analyzing the knowledge involved in assessing students' subtraction strategies. They affirmed that it is debatable whether the knowledge used by a teacher is exclusive to him or her, or is shared with other practitioners of mathematics. They focus discussion on certain cognitive dispositions and wonder who else, other than a mathematics teacher might have such kind of knowledge, thus moving the focal point of the debate from mathematics teacher knowledge to that of other professionals.

The answers we might gain from such comparisons (mathematics teachers vs. mathematicians, mathematics teachers vs. teachers of other subjects, etc.) are external to mathematics education as a discipline, in that they offer justifications that are recognizable and measurable but neither cognitive (concerning the processes involved in knowledge) nor epistemological (regarding the nature of knowledge). External referents (such as mathematicians) might provide useful markers for identifying static traits that differ from mathematics teachers such as the content of teacher knowledge, that is, what teachers' knowledge is about. However, they seem to be inappropriate in accounts of the complex dynamics of knowledge in use. Rather than framing the discussion of what makes mathematics teacher knowledge specialized in terms of external referents, we suggest an account of specialization understood in relation to mathematics teacher knowledge in action. That is to say, what makes mathematics teacher knowledge specialized is not so much "what" mathematics teachers know (which might indeed differ from other professionals), but "how" mathematics teachers know. This involves a shift away from the content of mathematics teacher knowledge to its usage and function, that is, how teacher knowledge comes into action (how it comes into being or how it actualizes). This shift in perspective foregrounds the context rather than the content.

Instead of an extrinsic perspective, we suggest taking an intrinsic view, that is, acknowledging the situatedness of mathematics teacher knowledge within the context of mathematics learning and teaching. Interestingly, Carrillo, Climent, Contreras and Muñoz-Catalán (2013) have already explicated a framework, termed the *Mathematics Teacher's Specialized Knowledge* (MTSK) framework, which is constructed on, and projects, an intrinsic perspective whereby the idea of specialization is framed with regard to the inseparability of knowledge and context. The key to recognizing and making visible what makes mathematics teacher knowledge specialized lies, we argue, in the context in which the knowledge comes into being. Contextuality, then, becomes the central concern. Obviously, that context matters is hardly new nor provocative (see e.g., Fennema & Franke, 1992); however, the way in which the term is commonly used differs from the point we want to advance in this paper.

In our view, whether knowledge is specialized or not is a question of whether the knowledge is contextually adaptive (Hashweh, 2005), that is, functional on a moment-by-moment basis, and highly sensitive to the changing details of the situation as teachers interact with the environment and with the students around them. This means, rather than expecting differences in knowledge (concerning quantity, quality, etc.) based on broad descriptions of context – such as school vs. scientific environment – the term "context" acquires a very different and deeper meaning than the ways it has been previously construed. This perspective assumes that context consists of situations and activities embedded in the learning-teaching complex in the immediate moment. In consequence, what signifies mathematics teacher knowledge might be better described (or can be better approached) from within the discipline. In this regard, mathematics teacher knowledge is treated not as static traits (that differ from other professions) but as interpretations of performances that are situated in the immediate context (see Brown, Danish, Levin, & diSessa, 2016). In this regard, Putnam and Borko (2000) argued that "professional knowledge is developed in context, stored together with characteristic features of classrooms and activities, organised around the tasks that teachers accomplish in classroom settings, and accessed for use in similar situations" (p. 13). As such, a mathematics teacher's action is not a simple display of a static system of some certain knowledge types, but rather a highly contingent and continually adaptive and proactive response that shapes, and is shaped by, the environment in which the teacher interacts.

In other words, it is not about *being* but about *becoming*, that is, it is less about static dispositions or traits differentiable from those of other professions and more about the complex dynamics of the usage and function of knowledge in context. Mathematics teacher knowledge becomes specialized in its adaptive function in response to the dynamics and complexities in which it comes into being.

From a Structuralist to an Anthropological-Sociocultural Approach

In this section, we adopt a critical stance to the disciplinary approach to teacher knowledge, an approach that is primarily structuralist in orientation and that argues from the viewpoint of teaching mathematics rather than from the standpoint of learning mathematics. We argue against a restrictive structuralist perspective that relies on, and projects, a reductionist understanding of knowing and learning, in which knowledge is construed as independent of the knower. Instead we argue for an anthropological-sociocultural perspective that accounts for the evolving nature of mathematical meaning in the learning process.

Shulman (1987) declared that subject matter knowledge per se "must be transformed in some manner if they are to be taught. To reason one's way through an act of teaching is to think one's way from the subject matter as understood by the teacher into the minds and motivations of learners" (p. 16). Generally speaking, the central task of teaching is considered as transforming subject matter knowledge into a form in which it is teachable to particular learners. This transformation of the subject matter is, according to Shulman (1987), heavily, if not wholly, determined by the disciplinary subject matter as the primary source of information for teaching and the principal route to informed decisions about instruction. Gudmundsdottir (1991) described this transformation as a "reorganization [of content knowledge] that derives from a disciplinary orientation" (p. 412) and Grossman et al. (1989) designated it as "translat[ing] knowledge of subject matter into instructional representations" (p. 32). As mentioned above, scholars in the field of mathematics education have recommended several discipline-specific practices of transformation that aim to unpack mathematics content in ways accessible to students: elementarizing, exemplifying, decompressing, and simplifying, among other. In this view, teachers must be able to take apart mathematical concepts, operations and strategies so as to enable students to gain access to the thought processes and ideas that they represent. Students, on the other hand, are considered as putting together the constituent pieces of those mathematical concepts, operation and strategies. Such assertions rely on, and project, a reductionist understanding of the knowing and learning processes; an understanding in which the knowing and learning processes are construed as putting together what teachers intentionally picked apart. This view not only distorts the complexity of the processes of knowing and learning mathematics, but also advocates the assumption that knowledge is independent of the knower.

Some general approaches in mathematics education have challenged reductionist views on knowing and learning, including, but not limited to, Gestaltism, constructivism, problem-solving, socio-culturalism, and complexity thinking. Here we follow anthropological-sociocultural perspectives, which, rather than consider knowledge as an object that exists apart from the individual, acknowledge the co-implicated nature of knowledge, knower and context. In this perspective, particular emphasis is given to the genesis of mathematical knowing and learning by accounting for historical and cognitive evolutions, dynamics, and changes. In this view, knowledge is considered a process rather than an object (see e.g., Radford, 2013) – to acknowledge the complex dynamics in knowing mathematics.

For instance, the *Didactic Mathematical Knowledge* (DMK) framework (Pino-Fan, Assis & Castro, 2015) is grounded in an onto-semiotic perspective of mathematical knowledge and instruction (Font, Godino & Gallardo, 2013; Godino, Batanero & Font, 2007). As such, the framework is rooted in anthropological-sociocultural assumptions about mathematical knowledge (where mathematics is understood as a human activity), and takes up the ontological assumption of a diversity of mathematical objects as well as the semiotic assumption of a plurality of languages and meanings. The DMK framework, similar to other proposals (e.g., Ernest, 1989), relies on, and projects, assumptions that transcends realistic-Platonic positions on the nature of mathematics and foregrounds an anthropological conception of mathematics. That is, teachers have to recognize the emergence of concepts, procedures, and propositions from mathematical practices, and attribute a central role to the various languages and artifacts involved in such practices. The applications – the use of mathematics as a cultural reality in itself to solve real-life or mathematical problems – promote a variety of meanings for mathematical objects, which must be progressively articulated in the learning process. Such a view acknowledges the embodied meanings of mathematical concepts that evolve in the learning process. The DMK framework particularly foregrounds an *epistemic facet* of teachers' didactical-mathematical knowledge which, according to Godino, Font, Wilhelmi and Lurduy (2011), interacts with other knowledge facets (affective, cognitive, ecological, interactional, and mediational). Consequently, the attentiveness (or mindfulness) to epistemological issues (such as the nature of mathematics and mathematics learning) is illuminated. From this perspective, teachers' sensitivity towards the epistemic genesis of mathematics and mathematics learning becomes a central aspect of what mathematics teacher knowledge signifies.

In short, an anthropological-sociocultural perspective acknowledges knowledge as an evolving process rather than a more or less static object that exists independent of the knower. In this view, not only the interaction between knowledge, knower, and context is highlighted, but also the historical and cognitive genesis of mathematical meanings. Thus, what makes mathematics teacher knowledge specialized is not the accumulation of dstinct facets of knowledge, but the teachers' stance towards knowledge, in the light of the historical and cognitive geneses of mathematical insights. This perspective calls for a shift in thinking about teachers' core tasks: the teachers' focus should not be on acting upon subject matter by re-structuring, re-interpreting, re-configuring, and re-building mathematical concepts to make them accessible to students, but instead on the complex interactions between students and subject matter. That is, the key is not teachers' capacity to unpack mathematics, but their capacity to unpack students' ways of understanding in order to make students' ways of mathematical thinking visible.

This is not to be understood as dichotomizing teachers' capacity for unpacking mathematics and their capacity for unpacking students' understandings, but to re-emphasize that teaching is not merely a top-down approach of transposing subject matter to the students but a bottom-up approach of students constructing mathematical ideas that are used as points of departure in the teaching-leaning complex.

From an Additive to a Transformative Approach

In this section, we adopt a critical stance to another apparently widespread tendency that seems to have implicitly driven recent discussions on teacher knowledge: the tendency towards atomizing teacher knowledge for the sake of accumulating distinct and refined dimensions of teacher knowledge. We argue for a transformative approach that goes beyond a merely incremental approach to facets of knowledge by turning back to Shulman's idea of blending knowledge facets.

The last three decades have been colored by various attempts to capture what mathematics teacher knowledge is about and what it entails. Research studies started out by distinguishing, refining, and adding to various dimensions of knowledge regarded as critical for teaching mathematics. Since then we have accumulated a considerable number of, often indistinguishable (see Silverman & Thompson, 2008), knowledge dimensions that, taken together, seem to provide a more refined picture of the multidimensionality of teacher knowledge. This undertaking allowed scholars to order, structure, and, most important, simplify the complexity of teacher knowledge, to reduce it to its observable and measurable parts.

The approach relies on the assumption that a full understanding of teacher knowledge should emerge from a detailed analysis of each of its parts. It is believed that the complexity of teacher knowledge can be studied by dissecting it into its smallest parts (knowledge facets, types, etc.), and that these knowledge units are the basis, or the fundamental particles, of what mathematics teacher knowledge signifies. Following these lines of thinking, reflections on mathematics teacher knowledge emphasize the nature of these parts – paying little attention to transformations that arise when knowledge elements are blended.

Instead of dividing and thinking in terms of multiple, distinct sub-categories of teacher knowledge, our disposition is to take a broader view that sees teacher knowledge as an organic whole.

Interestingly, Shulman (1987) already described PCK as "that special *amalgam* of content and pedagogy that is uniquely the province of teachers, their own special form of professional understanding" (p. 8, italics added). Here, Shulman understood PCK not as the summation or accumulation of content knowledge and pedagogical knowledge: "[…] just knowing the content well was really important, just knowing general pedagogy was really important and yet when you add the two together, you didn't get the teacher" (Shulman, cit. in Berry, Loughran, & van Driel, 2008, p. 1274). Rather, the amalgamation of content and pedagogy means "the *blending* of content and pedagogy" (Shulman, 1987, p. 8, italics added) into a new kind of knowledge that is distinctively and qualitatively different from the knowledge dimensions from which it was constructed. However, by proposing PCK as the amalgam of content and pedagogy without accounting for the complex interactions between these and other knowledge facets, Shulman left the task of further clarifying the blending process to other scholars.

Surprisingly, though many scholars paraphrased Shulman's idea of amalgamation, they almost always took the result of blending knowledge domains (that is, according to Shulman, PCK) as given and often considered it as static (for a critique, see Hashweh, 2005). In other words, many scholars ignored the complex dynamics of blending, a high interaction of knowledge facets that forms new structure not evident in the previous facets.

To the best of our knowledge, blending seems to be an undertheorized phenomenon in research on teacher knowledge. Recently, Scheiner (2015) has suggested construing teacher knowledge as a complex, dynamic system of various knowledge atoms, which are understood as blends of different knowledge facets. The idea of 'knowledge atom' shares similarities with Sherin's (2002) idea of 'content knowledge complexes' construed as "tightly integrated structures containing [pieces of] both subject matter knowledge and pedagogical content knowledge" (p. 125) repeatedly accessed during instruction. Scheiner (2015) proposed that teacher knowledge is dynamic not simply because it evolves dynamically (which it does), but because it forms dynamically: teacher knowledge is dynamically emergent from the interactions of knowledge facets. This interaction of knowledge facets is in the nature of what Fauconnier and Turner (2002) described as *conceptual blending*. In technical terms, blending is a process of conceptual mapping and integration, a mental operation for combining frames or models in integration networks that leads to new meaning, global insights, and conceptual compression (see Fauconnier & Turner, 2002). The essence of conceptual blending is to construct a partial match, called cross-space mapping, between frames from established domains (known as *inputs*), to project selectively from those inputs into a novel hybrid frame (a *blend* or *blended model*), comprised of structure from each of its inputs, as well as a unique structure of its own (*emergent* structure). Crucially, the inputs are not just projected wholesale into the blend, but a combination of the processes of projection, completion, and elaboration (or 'running' the blend) "develops emergent structure that is not in the inputs" (Fauconnier & Turner, 2002, p. 42). The point we want to make here is that knowledge facets interact dynamically to form emergent structures. Not only do new elements arise in the blend that are not evident in either input domain on its own, but blending accounts also for the interdependencies of knowledge dimensions: the production of a blend is recursive, in the sense, that blends depend on previous blends.

Scheiner's (2015) proposal of teacher knowledge as a complex, dynamic system of various knowledge atoms attempts a dialectic between atomistic and holistic views of teacher knowledge. It puts the refinements of teacher knowledge identified and gained through atomistic approaches together into a complex system of blends that – as a whole – is more than the sum of its parts.

In a nutshell, a complex system perspective regards teacher knowledge as dynamically emergent and dimensions of teacher knowledge as being organically interrelated. It emphasizes that various knowledge facets are in constant dialogue with each other, inform each other, and interact dynamically to form emergent structures. Thus, the key relies not on accumulating types of teacher knowledge but on blending knowledge facets that emerge dynamically. Accumulating teacher knowledge facets is additive (or complementary), but blending is transformative.

Discussion

In the three previous sections, we have critically appraised what the current literature implicitly represents as making mathematics teacher knowledge specialized. In each section, we have tried to make explicit the more serious limitations of the grounds on which at least three general tendencies stand, and which seem to have been inherent in the current discussion on teacher knowledge. Each section provides provocative new ways of thinking about the issue under consideration.

First, we called for an account of specialization that comes from the inside rather that the outside (such as comparisons with professionals working in other disciplines). In recognizing the situated nature of mathematics teacher knowledge in the immediate context, the complex dynamics of the usage and function of knowledge in the immediate context can be underlined. As such, specialization is not a state of being but a process of becoming: mathematics teacher knowledge becomes specialized in its adaptive function in response to the dynamics and complexities in which it comes into being.

Second, we argued that an account of specialization cannot be provided with itemisation of mathematics teacher knowledge, but rather through teachers' epistemological stance toward knowledge and the sensitivity for the historical and cognitive geneses of mathematical insights. Going beyond a structuralist viewpoint, in which the teacher's task is considered to be unpacking the subject matter of mathematics, we encouraged the view of teachers unpacking students' understandings to make students' ways of mathematical thinking explicit.

Third, we argued that an account of specialization lies not in the sum of the parts of mathematics teacher knowledge but in its organic whole, that is, various knowledge facets constantly in dialogue with each other, informing each other, and interacting dynamically to form emergent structures. We proposed a complex system perspective that construes mathematics teacher knowledge as blends of various knowledge facets that emerge dynamic structure.

On the one hand, these alternative views point to several aspects that scholars attempted to encompass in their use of the notion of *knowing* rather than *knowledge*: knowledge is usually treated as static, explicit, and objective, whereas what is described as knowing is seen as dynamic, tacit, and contextualized (see Adler, 1998; Ponte, 1994). However, the alternative views outlined above foreground aspects that might contribute further to the discussion of knowledge versus knowing. First, whereas knowledge has been debated as either existing independently of the knower (the realist viewpoint) or only existing in the mind of the knower (the relativist viewpoint), with the term knowing we can signal the inseparability of knowledge and knower. That is, it makes no sense to talk about something being known without also talking about who knows it (and under which circumstances). Second, what is called knowledge is usually perceived as a state of being (or product), whereas what is described as knowing is seen as an emergent process – a process of becoming. However, this is not a call for a distinction between product and process, since the main point is seen in the complex dynamics underpinning the stability of established knowledge (see Davis & Simmt, 2006). It implies the dynamic character of knower, knowledge, and context such that all three are changing and evolving over time. This means knowing is not just situated in place – that is, it is contextual and embedded in the practices of teaching (Adler, 1998) – but also situated with respect to time and other factors, given that the context of knowing is similarly dynamic and changing over time. That knowing is situated with regard to time, place and other factors implies that it cannot be reduced to some observable and measurable by-products. The whole venture is to understand mathematics teacher knowing as it is, as it comes into being, as it works in the immediate context; that is, to take a holistic (rather than a reductionist) view that acknowledges mathematics teacher knowing as highly personal, embodied, enacted, and performed. Any approach toward what makes teacher knowledge specialized must deal with this complex whole rather than with piecemeal facets or types of knowledge (see Beswick, Callingham, & Watson, 2012).[3] Of course, such sensibilities are not entirely new. They might be argued to have been represented in the discourses of different movements of thought such as cognitive approaches and situated approaches (see Kaiser et al., 2017), as well as other discourses. However, the view advanced here takes the discussion to realms that often cast knowing and knowledge as oppositional.

On the other hand, and more importantly, the alternative viewpoints converge on the understanding that it is not a kind of knowledge but a *style of knowing* that accounts for specialization in mathematics teacher knowledge. To elaborate this aspect in more detail: In the past, the focus was primarily on knowledge about/of/for/in the discipline. This resulted in multiple descriptions and distinctions, such as knowledge about mathematics versus knowledge of mathematics, or mathematical knowledge for teaching as opposed to mathematical knowledge in teaching, and knowledge for teaching mathematics in contradistinction to knowledge in teaching mathematics, all primarily concerned with the question of ‘what’ mathematics teachers know. In this regard, comparisons such as mathematics teachers versus mathematicians or mathematics teachers versus teachers of other subjects were assumed to be decisive, as it was believed that it was the kind of knowledge – whether quantitatively or qualitatively different – that set mathematics teachers apart from other professionals. However, the alternative views discussed above consider the yet unsettled question of ‘how’ teachers knowing comes into being rather than pointing to the question of ‘what’ teachers know. This brings to the fore the complex, dynamic usage, function, and interaction of mathematics teacher knowing, a position that goes beyond accounts that primarily address kinds of teacher knowledge. We intend to enunciate this shift in perspective by calling for attention to mathematics teachers’ *styles of knowing* rather than merely teachers’ *kinds of knowledge*. We believe that this shift in perspective is critical as it provides a new light on the discussion of the nature of mathematics teacher knowledge that allows us to better integrate knowledge and action. It articulates mathematics teacher knowledge more as a mindset rather than as some static traits or dispositions. To cast this idea in a term, we suggest a fine distinction in thinking about the issues under consideration: knowledge about/of/for/in a discipline and *disciplinary knowing*. Knowledge about/of/for/in the discipline prompts the question of different *kinds of knowledge*, while disciplinary knowing prommpts the question of a *style of knowing* that is a function of particular activities, orientations, and dynamics recognizably disciplinary. From this perspective, we argue that it is *mathematics educational knowing* that signifies specialization in mathematics teacher knowledge.

Conclusions

Mathematics teacher knowing is a mysterious phenomenon indeed. To acknowledge this mystery is not to mystify mathematics teacher knowing, but to express our recognition of the exquisite complexity of how mathematics teacher knowing comes into being. Breaking up the complex nature of teacher knowledge for the sake of insights leads to atomizing our understanding, our thinking, of what makes mathematics teacher knowledge specialized. Such insights are themselves fragmented, not holistic. The piecemeal, atomistic, analytic approach (as advocated in the past) does not work in relation to the complex usage, function, and interaction of teacher knowing. Any approach toward what makes teacher knowledge specialized must deal with the complex whole rather than with some piecemeal facets or types of teacher knowledge.

In this paper, new avenues for theoretical reflection on some of the major orientations and tendencies in the field of mathematics teacher knowledge were outlined. These reflections were not intended to exhaust the object of consideration, but to include those approaches, initiatives, and theoretical insights that might prompt re-thinking about what mathematics teacher knowledge signifies.

We explained that the question of what makes teacher knowledge specialized cannot be comprehensively answered by only addressing “what” teachers know, but we need to account for “how” teachers knowing comes into being. The alternative views discussed in the paper bring to the foreground that it is not a kind of knowledge but a style of knowing that accounts for specialization in mathematics teacher knowledge. Such style of knowing is not a state of being but a process of becoming – the becoming of a mathematics educational mindset.

This call for a style of knowing is rather different from what normally receives emphasis in discussion of mathematics teacher knowledge. We hypothesize that considering specialization as a style of knowing (rather than a kind of knowledge) can have far-reaching consequences not only for conceptualizing mathematics teacher knowledge.

With respect to mathematics teacher education, for instance, considering specialization as a style of knowing (rather than a kind of knowledge) advocates a holistic approach to mathematics teacher education programs, criticizing the separate acquirement of different kinds of knowledge (generally acquired from different academic departments such as mathematics, education, psychology, among others). Mathematics teacher education programs should be deliberately designed in an integrated fashion to support teachers in blending insights from various disciplines including, but not limited to, mathematics, education, and psychology, thereby creating novel styles of knowing that empowers teachers to reshape the way they view their own profession. It is reasonable to assume that such styles of knowing develop gradually, rooted in authentic activities and within a community of individuals engaged in inquiry and practice (see Putnam & Borko, 2000). Further, a shift toward a style of knowing is expected to affect researchers’ and educators’ perceptions of teachers’ professional identity, as the path to a mathematics educational mindset is a journey, not a proclamation. This would mean giving up deficit-oriented discussions on teacher knowledge in terms of identifying and fixing teachers’ lack of knowledge (Askew, 2008). The central concern for future research, then, is to understand those mindsets, which underpin any authentic form of mathematics educational knowing. It is hoped that this call for a style of knowing offers a new vision of what makes mathematics teacher knowledge specialized.

Notes

[1] We prefer using the term ‘specialized’ instead of ‘special’ with respect to mathematics teacher knowledge. The latter implies the assertion of a quality of teacher knowledge that is distinguishable from something. We use the term ‘specialized’ to indicate a quality of mathematics teacher knowledge that comes into being when enacted.

[2] We use the term structuralism (or structuralist) in a broad sense as described by Bourdieu (1989): “By structuralism or structuralist, I mean that there exist, within the social world itself and not only within symbolic systems (language, myths, etc.), objective structures independent of the consciousness and will of agents, which are capable of guiding and constraining their practices or their representations” (p. 14).

[3] Notice that we do not construe the relationship between knowing and knowledge as contradictory but rather as dialectical. In terms of the onto-semiotic approach there is no mathematical practice without objects, or objects without practice, which is equivalent to the issues of knowing and knowledge discussed here.

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