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INSTITUTIONAL AND PERSONAL MEANINGS OF MATHEMATICAL PROOF

ABSTRACT. Although studies on students' difficulties in producing mathematical proofs have been carried out in different countries, few research workers have focussed their attention on the identification of mathematical proof schemes in university students. This information is potentially useful for secondary school teachers and university lecturers. In this article, we study mathematical proof schemes of students starting their studies at the University of Córdoba (Spain) and we relate these schemes to the meanings of mathematical proof in different institutional contexts: daily life, experimental sciences, professional mathematics, logic and foundations of mathematics. The main conclusion of our research is the difficulty of the deductive mathematical proof for these students. Moreover, we suggest that the different institutional meanings of proof might help to explain this difficulty.

1. Introduction

There has been a growing interest, in mathematics education, in the problems of teaching and learning of proof. One indicator of this concern is the electronic newsletter about proof edited by M. A. Mariotti (URL: http://www-didactique.imag.fr/preuve). This interest is justified by the essential role of validation within mathematics, and by the students' low level in understanding and building mathematical proofs (Galbraith, 1981; Fischbein, 1982; Senk, 1985; Martin and Harel, 1989; Chazan, 1993; Battista and Clements, 1995; Zaslavsky and Ron, 1998; Healy and Hoyles, 2000; Recio, 2000).

Although the aforementioned investigations have made important contributions to mathematics education, there is still some room for new research that clarifies the meaning of mathematical proof, its different types and mutual relationships. The idea of proof, which is understood in a rigid and absolute way by the mathematical community, seems to have been considered the only valid conception. However, we consider it necessary to carry out a systematic study of the various meanings of proof, not just from the subjective point of view, but also in different institutional contexts. Such study would serve to compare the contributions from different research works, pose new research problems, give alternative interpretations to students' observed difficulties, and elaborate new didactic proposals.

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In this article we present the results of a research study about students' capability to build elementary deductive proofs when they start their university studies. We interpret and classify the students' answers to a written questionnaire (including a geometrical and an arithmetical problem) as personal proof schemes (Harel and Sowder, 1998), and we identify four basic types of proof schemes. A quantitative analysis of the results reveals the low level of this capability in our students (University of Córdoba, Spain).

Starting from an anthropological view of knowledge, we put forward the hypothesis that the main features of students' proof schemes can be related with the meanings of proof in different institutional contexts, and that such institutional meanings could offer explanations of the personal proof schemes. This statement is supported by our study of the institutional meanings of proof (Godino and Batanero, 1998) in the following contexts: daily life, experimental sciences, professional mathematics, logic and foundations of mathematics.

We propose that the teaching of proof in school mathematics should take into account these diverse institutional meanings in the aim of helping students discern circumstances appropriate for each type of argument.

2. Experimental research on students' proof schemes

In the aim of characterizing mathematical proof schemes of students entering the University of Córdoba (Spain), a written questionnaire was designed and given to the students in the course of the academic year 1994–95. Later on, in 1997–98, the same written questionnaire was given to a second sample of students, in order to verify our previous results.

2.1. *Sample*

The questionnaire was given to the students just a few days after the classes at the University of Córdoba (Spain) had started. In the first sample, which was taken at the beginning of the academic year 1994–95, 429 students who took a mathematics subject in different faculties and polytechnic schools, were selected. At the beginning of the academic year 1997–98, the test was given to a second sample of 193 students from the same university institutions.

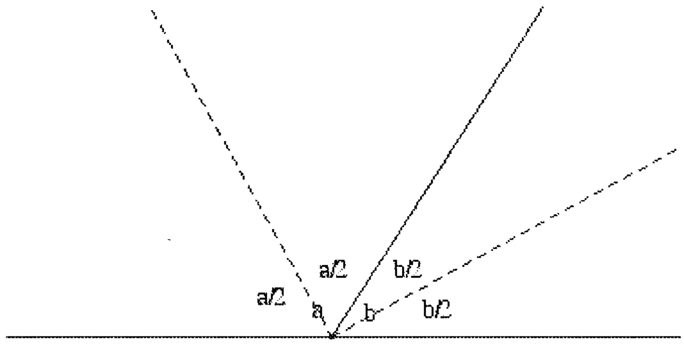

Figure 1. Drawing related to the expected solution of Problem 2.

2.2. *Questionnaire*

The written questionnaire was composed of two problems, whose solution required the students to possess some proving capability in mathematics. The problems involved very elementary notions, well known by all students. The first problem was stated as follows:

Problem 1 (Arithmetic)
Prove that the difference between the squares of every two consecutive natural numbers is always an odd number, and that it is equal to the sum of these numbers (Recall that the set of natural numbers is the infinite series of numbers 0, 1, 2, 3, ...).

This problem was intended to identify students' mathematical proof schemes in a numerical context. The standard correct answer is as follows:

Let us call n and n+1 the two consecutive natural numbers. Then the difference of squares of these numbers is: $(n+1)^2 - n^2 = n^2 + 2n + 1 - n^2 = 2n + 1$. *Therefore:* $(n+1)^2 - n^2 = 2n + 1 = n + (n+1)$. *The first equality proves that the difference between the squares of every two consecutive natural numbers is an odd number (an even number plus one). The second equality proves that this odd number is equal to the sum of these two consecutive numbers.*

The second problem was the following:

Problem 2 (Geometry)
Prove that the bisectors of any two adjacent angles form a right angle. (Recall that two angles are adjacent if they have the vertex and a side in common, and their sum is a flat angle, that is to say, 180°. Recall that a right angle measures 90°. The angle bisector is the ray that splits the angle into two equals parts).

This problem was intended to identify students' mathematical proof schemes in a geometric context. The standard correct answer is as follows:

If a and b are two adjacent angles, then:
$a + b = 180°$

Therefore, the angle that the bisectors form is:
a/2 + b/2 = (a+b)/2 = 180°/2 = 90°

2.3. *Categorisation of answers*

Students' answers were classified into five different categories. These categories were elaborated in a previous study on other university students' generalization and symbolization abilities (Recio and Godino, 1996):

1. The answer is very deficient (confused, incoherent).
2. The student checks the proposition with examples, without serious mistakes.
3. The student checks the proposition with examples, and asserts its general validity.
4. The student justifies the validity of the proposition, by using other well-known theorems or propositions, by means of partially correct procedures.
5. The student gives a substantially correct proof, which includes an appropriate symbolisation.

Answers of Type 1 include basic difficulties in understanding the statement of the problem, in making abstract operations, etc. To identify clearly the type of difficulty in each case it would be necessary to carry out clinical interviews. Answers of Type 2 are easy to interpret, so it is not necessary to give illustrative examples thereof.

We can select the following examples of answers of Type 3 in the arithmetic problem:

a) The student checks the proposition with examples and generalises its validity by asserting that it is always true:

   36 – 25 = 11
   9 – 4 = 5
   It is true that the difference between the squares of any two consecutive natural numbers is always an odd number, equal to the sum of these numbers.

b) The student checks the proposition in particular cases, introducing some incorrect or very insufficient symbolic formulation:

   36 – 25 = 6 + 5
   11 = 11
   We have
   49 – 36 = 7 + 6
   13 = 13
   We have
   $A^2 - B^2 = A + B$

The use of symbolism indicates that there is some generalisation involved.

c) The student checks the proposition in particular cases, and also develops a generalising logical argument, which is clearly incorrect or insufficient:

It will always be an odd number because the sum of an even number and an odd number will always be an odd number, and in two consecutive numbers one has to be even and the other odd.

In this case, the concepts used in the argumentation already have some generality. The student considers the objects not as specific entities, but as representatives of categories, although the reasoning is still only just beginning.

In the geometrical problem we can select the following examples of Type 3 answers:

a) The student checks the proposition with examples and generalises its validity by asserting that it is always true:

The simplest case is that of two right adjacent (90°) angles. If we draw their bisectors a and b, the angles are divided into angles of 45° and their sum is 90°, that is to say, a right angle. The same thing would happen for any given angle. If we draw the bisectors of any two adjacent angles, these will form an angle of 90°. Another example of adjacent angles would be 30° and 150°... (The student continues describing the example).

b) The student checks the proposition in particular cases, introducing some wrong or very insufficient symbolic formulation:

Let the two angles be A and B, A + B = 180°. Let A = 30° and B = 150°. The bisectors will be 15° and 75°; 15° + 75° = 90°. The bisectors give rise to an angle of 90°.

c) The student checks the proposition in particular cases, and also develops a generalising logical argument, which is clearly incorrect or insufficient:

If each angle is 90°, the bisectors are 45°, so that 45° + 45° = 90°, which means that the bisectors make an angle of 90°. If the angles have a value other than 90°, one increases as much as the other decreases and the bisectors still make an angle of 90°.

A general feature of Type 4 answers is that the validity of the proposition is logically justified, in a partially correct way, based on other well-known theorems or propositions.

In the arithmetic problem we can select the following examples:

a) The student gives a logical, not symbolic, and substantially correct argument (to the extent that the lack of symbols allows it):

The square of an odd number is an odd number. The square of an even number is an even number. The difference between an even number and an odd number (or between an odd number and a even number) is an odd number. Given two consecutive natural numbers one is an even number and the other an odd number; hence their squares will be an even number and the other an odd number, and the difference of their squares will be an odd number.

b) The student gives a symbolic, partially incorrect argument:

$x^2 - (x + 1)^2 = y; y = x + (x + 1)$
$x^2 - (x^2 + 2x + 1) = x + (x + 1)$
$x^2 - x^2 - 2x - 1 = x + x + 1$
$-2x - 1 = 2x + 1$
$-4x = -2$
$4x = 2$
$x = 2; x + 1 = 2 + 1 = 3$
$y = 2 + (2 + 1) = 5$

The student began the argumentation correctly, but he was unable to give a complete correct proof. Anyway, he operates with symbols.

The following are examples of Type 4 answers in the geometric problem:

a) The student gives a logical, not symbolic, and substantially correct argument (to the extent that the lack of symbols allows it):

If two angles are adjacent, their sum is 180° and when you make the bisectors, that is to say, when you split each of the two angles, the sum of half one angle and half the other is 90°. That is the same as splitting the straight angle into its two halves.

b) The student gives a symbolic, partially incorrect argument:

Suppose we have a 50° angle and a 130° angle. Their sum is 180°. If the bisector splits the angle in two equal parts, each part is 25° and 65°, respectively. Therefore, the sum of these resulting angles is 90°, a right angle (a drawing is included).*We have:*
$g = 180°; a = 50°; b = 130°; f = 90°$
$g = a + b$
$a/2 + b/2 = g/2 = f$
$g/2 = f.$

2.4. *Quantitative results*

In Table I we present the absolute frequencies and percentages of each type of answers in the two problems included in the questionnaire.

We can observe that the percentage of students giving a substantially correct mathematical proof to each problem is less than 50%. This percentage is reduced to 32.9% of students when we consider correct answers to the two problems, since only 141 of 429 students solved both problems

TABLE I

Frequencies and percentages of types of answers in the 1994–95 population sample

Type of answer	Problem 1 (Arithmetic)			Problem 2 (Geometry)		
	Frequency	%	Cumulative %	Frequency	%	Cumulative %
1	16	3.7	3.7	34	7.9	7.9
2	48	11.2	14.9	85	19.8	27.7
3	122	28.4	43.4	75	17.5	45.2
4	39	9.1	52.5	53	12.4	57.6
5	204	47.5	100	182	42.4	100

correctly in the academic year 1994–95. Only 44 of 193 students gave correct answers to both problems in the 1997–98 academic year (i.e., 22.8% of the total number of students).

These results, and other complementary data (Recio, 2000), confirm our initial assumption about the great difficulties that deductive mathematical proof present for students starting university, even in the case of quite elementary propositions.

In order to analyse the possible dependence of the test results on the mathematical content of the problems (arithmetical vs. geometrical), we also studied the association between the test scores in both problems. An association between these scores would suggest that the students' mathematical proof level is high or low, regardless of the problem's mathematical content, while lack of association would indicate an influence of this content in the proof schemes. To study this association we prepared the cross tabulation of the variables (ARITHMETIC, GEOMETRY) in Table II.

In Table II the relative frequency of each cell refers to the column total, i.e., it is the relative frequency distribution of the score on the arithmetic problem conditioned by the score on the geometric problem. We can observe the increase of one score as a function of the other, as well as the high dependence of the conditioned distributions of rows regarding columns.

The frequencies tend to concentrate in general on the diagonal and adjacent lines; this fact is quantified with the Goodman and Kruskal' Gamma association coefficient for ordinal variables equal to 0.71, which is relatively high. This is a measure of association for rxc contingency tables of ordinal variables. It measures, on a (–1, 1) scale, the degree of agreement between two different orderings of the same objects. In this case it measures the degree of agreement between the scores assigned to each student in the two different problems, which was, therefore 71%. Under the conditions of the test, this suggests that the mathematical content (arithmetical/

TABLE II

Frequencies of the two-dimensional variable (ARITHMETIC, GEOMETRY) in the 1994–95 population sample

	Score in the geometry problem					
	1	*2*	*3*	*4*	*5*	Total
Score in the arithmetic problem						
1	**7**	**9**	**0**	**0**	**0**	**16**
	20.6	10.6	0.0	0.0	0.0	3.7
2	**9**	**26**	**10**	**2**	**1**	**48**
	26.5	30.6	13.3	3.8	0.6	11.2
3	**16**	**30**	**33**	**18**	**25**	**122**
	47.1	35.3	44.0	33.9	13.7	28.4
4	**1**	**8**	**5**	**0**	**15**	**39**
	2.9	9.4	6.7	18.8	8.2	9.1
5	**1**	**12**	**27**	**23**	**141**	**204**
	2.9	14.1	36.0	43.4	77.5	47.5
Total	**34**	**85**	**75**	**53**	**182**	**429**
	7.9	19.8	17.5	12.4	42.4	100.0

geometrical) of the problems had little influence on mathematical proof capacity, showing that students' proof schemes stayed relatively independent from the mathematical content of the problems posed.

2.5. *Interpretation of answers as proof schemes*

The high association value found between students' answers to the arithmetic and the geometry problems led us to interpreting the proposed categories as personal schemes of mathematical proof. We considered these schemes as the subjects' stable answer models in proof problems with elementary content and structure, where knowledge of certain techniques of symbolization and generalization was needed.

We do not take into account Type 1 answers to classify the students' proof schemes, since the analysis of these answers would require an in-depth study using clinical interviews. This option does not affect the validity of the proposed model.

We can classify Type 2 answers, which are mere confirmations of the propositions to prove, using particular examples, as explanatory argumentations. In these processes the subject explains, by means of specific examples, the meaning of the proposition to prove. There is neither a true intention to validate the proposition, nor an intention to affirm the validity of the proposition for all possible cases. Since there is only an explanatory intention, we describe these types of answers as *explanatory argumentative schemes*.

Type 3 answers are based on verifying the propositions given to prove by using particular examples, without the intention of justifying the general validity of the proposition and using empirical-inductive procedures. Therefore, we refer to these types of answers as *empirical-inductive proof schemes*.

Type 4 answers develop informal logical approaches, based on the use of analogies, graphical tools, etc. We classify these types of answers as *informal deductive proof schemes*.

Type 5 answers are elementary forms of deductive proofs. This elementary character is imposed by the simplicity of the problems that are posed. However, the answers follow a formal approach, more in agreement with the transformation rules of a symbolic and algebraic language, in which the mathematical terms operate, than to the specific meaning of these terms. That is why we call them *formal deductive proof schemes*.

It is necessary to point out that students might use different schemes when solving more complex problems. We have checked this fact in a qualitative research carried out in the course of the 2000–2001 academic year. In this study, we posed the following problem to a sample of university students: *Prove that the sum of the interior angles of a triangle is 180°*. We found that the same student could start with an empirical-inductive procedure and end up using a more or less formal deductive scheme.

We also observed that the lecturer's instruction was effective in developing formal deductive proof schemes in the students, although empirical inductive schemes remained and were resistant to change. Empirical inductive schemes were the spontaneous type of argumentation in a high percentage of students when they were confronted with new problems, in which it was necessary to develop new proof strategies, different from the learned formal procedures.

3. INSTITUTIONAL MEANINGS OF MATHEMATICAL PROOF

In this section we analyse some institutional meanings of mathematical proof, as a preliminary step towards the study of relationships between the institutional meanings of proof and students' mathematical proof schemes.

From a cultural viewpoint, Wilder (1981) wrote that, "we must not forget that what constitutes 'proof' varies from culture to culture, as well as from age to age" (p. 346). We will try to show that this relativity should be extended to the different institutional contexts, when we are interested in the psychological and didactic problems involved in the teaching of proof.

In this section we study the diverse meanings of proofs in the following institutional contexts: daily life, empirical sciences, professional mathematics, logic and foundations of mathematics. We recognise that it is also possible to identify more local viewpoints where the problem of truth and proof takes on specific connotations within these contexts. For example, the main tendencies in the foundations of mathematics (logicism, formalism, intuitionism, and quasi-empiricism) hold different views about the role of mathematical proof and the criteria for its validity (Hanna, 1995, p. 42). However, we consider the level of analysis adopted in this article sufficient to show the diversity of identifiable 'proof objects', and the variety of theory and practice firmly established with regard to mathematical proof.

3.1. *Daily life*

In daily life, people normally use informal argumentations, which are situational, depending on the context and even on the subject's own emotional situation (Miller-Jones, 1991). This type of informal argumentation does not necessarily produce truth, since it is based on local value considerations, which lack the objective features of proof. The statement has no absolute and universal validity; an example not following the rule will not completely invalidate the rule, although the validity is increased when more facts satisfying the statement are found.

According to Fernández and Carretero (1995, pp. 41–43) the main features of this type of argumentation are:

a) It is applied to issues relevant to the person who makes the argumentation.
b) This argumentation is very dynamic and dependent on the situational context.
c) It is applied to open, fuzzy and not deductive tasks.
d) It uses the daily life language, instead of a formal and symbolic language.
e) It is used in all knowledge domains, even in mathematics and scientific problems.

Students may have difficulties in distinguishing the *intuitive argumentation* they use in their daily life from the deductive reasoning required in the mathematics classroom.

Polyá (1953) studied intuitive reasoning in mathematics, considering it to be the reasoning we use to formulate our mathematical conjectures and calling it *plausible reasoning*. Garuti, Boero and Lemut (1998, p. 345) affirm that there is a cognitive continuity between the production of conjectures and the construction of proofs; therefore, informal mathematical arguments might constitute the first levels of mathematical proof.

3.2. *Experimental sciences*

In contrast with natural argumentation in daily life, scientific argumentation has a validating intention, which leads it to generate scientific knowledge, that is, rational and objective knowledge, conditioned by experimental confirmation.

Scientific theories are objective models meant to represent reality. Scientific models appear when we detach ourselves from our daily life and introduce objective criteria of experimental checking. According to Fourez (1994) the essential nature of scientific argumentation and what distinguishes it from the daily life argumentation is the need for experimental proof.

Intuitive argumentation of daily life is replaced by experimental proof; beliefs are replaced by theories, which are experimentally validated. According to Popper (1972), theories develop permanently and progress by falsification of previous hypotheses.

Scientific theories are formulated using high-level mathematical languages; mathematics appears as a tool for expressing scientific facts and mathematics argumentation takes some connotation of experimental proof in scientific contexts.

Mathematical theories are considered to be true because they can be proven experimentally, in different phenomenological situations, regardless of their formal and deductive interrelations. For some authors, for example Kline (1980), mathematical truth should be evaluated by its applicability to the physical world; mathematics is true because it works, and when it does not work it should be modified. According to this interpretation, the usefulness in founding consolidated scientific theories is what proves, in the end, the validity of mathematical theories.

Scientific proof, based in experimental verification, is introduced in mathematics and induces a specific way of argumentation, which we call *empirical-inductive proof.* This is a first validating step, where some particular cases of the proposition to be proved are experimentally verified. This validation must later be complemented using deductive methods, but it gives certain logical consistency to conjectures made by way of intuitive procedures.

3.3. *Professional mathematics*

The argumentative process that mathematicians develop to justify the truth of mathematical propositions, which is essentially a logical process, is called mathematical proof.

Deductive proof is the prototypical pattern of mathematical proof. For Dieudonné (1987), validating rigor is linked to axioms and formalisation. Nevertheless, this formalist rigor decreases in practice. Formalised proofs become extraordinarily complex. Livingston (1987), for example, has shown the complexity of proving the uniqueness of the identity element in an algebraic group (e = e * e' = e'), as compared to its triviality when using informal argumentation. In Resnick's opinion (1992), this explains why contemporary mathematics is full of working proofs, that is, informal proofs. Moreover, new strategies for mathematical validation are arising from mathematics itself, which challenge the classical conception of deductive 'line by line' proof, such as zero-knowledge proof, holographic proof, visual proof, and, in general, proofs based on experimental confirmations (Hanna, 1995, p. 43). These proofs are based mainly on computer programs, and incorporate random validation procedures. As Hanna pointed out (1989, p. 20), the traditional formalist conception, based on an abstract and rigorous view of mathematics, is changing: "In the last two decades several mathematicians and mathematical educators have challenged the tenet that the most significant aspect of mathematics is reasoning by deduction, culminating in formal proofs. In their view there is much more to mathematics than formal systems. This view recognises the realities of the mathematical practice. Mathematicians admit that their proofs can have different degrees of formal validity – and still gain the same degree of acceptance. Mathematicians agree, furthermore, that when a proof is valid by virtue of its form only, without regard to its content, it is likely to add very little to the understanding of its subject and, ironically, may not even be very convincing".

3.4. *Logic and foundations of mathematics*

In logic and foundations of mathematics the notion of proof appears linked to deduction and formal systems. Logical argumentation is essentially a deductive argumentation. Pure deductive argumentations take place in axiomatic and formal systems. According to Garnier and Garnier and Taylor (1996), an axiomatic system is,

a) a collection of indefinite terms and symbols;
b) syntactic rules to build sentences and formulas starting from symbols and indefinite terms;

c) a collection of correctly built sentences, called axioms.

The inference rules determine how the sentences representing theorems can be deduced from axioms. In a mathematical theory proof is a sequence of propositions, each of which is an axiom or a proposition that has been derived from axioms by inference rules. A theorem is a proposition obtained this way using a proof. Formalisation replaces a semantic conception of truth – as adaptation between thought and external reality – by a syntactic conception of truth, interpreted as coherence within a certain formal system.

The meta-language of a system is the language to describe the system, to speak of it and investigate its properties. An important meta-linguistic property is consistency. A system is consistent if it is free from contradictions. A fundamental meta-mathematical result is Gödel's incompleteness theorem, which, according to Kline (1980), can be formulated saying that if a formal theory *T* containing the whole number theory is consistent, then it is incomplete. This means that there is at least one proposition of number theory, which we can call *S*, such that neither *S* nor non-*S* are demonstrable in the theory.

Gödel's incompleteness theorem proves that for any mathematical theory containing elementary arithmetic, if it is consistent, it is not complete; hence it cannot contain all mathematical truth. That means, finally, that mathematics cannot be limited to merely a formal system. Consequently, mathematical truth loses its character of absolute necessity, presenting a pragmatic value. We cannot automatically decide if an informal argumentation is correct or not, by a process of formal derivation, but this is rather a question of agreement between parts, using subjective considerations of people taking part in the proof process.

3.5. *A pragmatic view of proof*

There are certainly some common features in the uses of the word 'proof' in the different institutional contexts described, and this allows us to consider proof in a general sense. But this generic, abstract, metaphysical way of thinking should not conceal the rich and complex variety of meanings acquired by the concept of proof, or, better, by the diversity of 'proof objects' each of which is given a local meaning by the members of such institutions. We believe it is interesting to consider not just one, but several concepts of proof depending on the subjective and epistemological viewpoint, when we are interested in the psychological and didactic problems involved in the processes of validating mathematical propositions (Godino and Batanero, 1998; Godino and Recio, 1997).

By recognising this diversity of objects and meanings, we shall be in a better position to study the components of meaning, the circumstances of their development, the roles performed in the different contexts. In fact, we would thus understand better the ecological relationships established between the objects and the systemic nature of their meaning. This onto-semantic modelization can help us take into account the cognitive conflicts faced by anybody who is forced to participate as a subject in different institutional contexts.

4. Relationships between students' proof schemes and institutional meanings of mathematical proof

One of the main current trends in mathematics education looks at the processes of teaching and learning from a socio-cultural perspective and interprets mathematical objects as cultural entities, which are socially shared. The key idea is that the individual and the social domains of mathematical knowledge are interrelated. Ernest (1994) considers these two domains intrinsically linked: apart from individual and idiosyncratic processes, people learn and build knowledge, while interacting with each other.

From an anthropological perspective, Chevallard (1992) and Godino and Batanero (1998) consider that mathematical knowledge is developed within institutions and hence it must be considered as a socio-cultural product. Individuals are always members of several institutions and they have to share their collective ways of thinking and reasoning; their experiences are conditioned by the institutional context, its language and type of social interactions.

According to this epistemological and socio-cultural framework, students' mathematical proof schemes should be related to the institutional meanings of proof. This relationship can be considered as a two-way influence: i) personal schemes can be influenced by the meaning of proof in the institutions, of which the students are members; ii) additionally, the institutional meanings of mathematical proof emerge from the personal schemes prevailing in these institutions.

Since students are simultaneously subjects of different institutions (daily life, experimental science classes, mathematics classes, philosophy and logic classes, etc.), where different argumentative schemes are carried out, it is reasonable to expect that they should have difficulties in discriminating between the respective uses of each type of argumentation. Consequently, we consider such institutional meanings of proof to be explanatory factors for the subjective schemes, and therefore we suggest that they should be taken into account and investigated in depth.

We consider that the *personal explanatory argumentation schemes* could correspond to elementary intuitive argumentations, almost without validating intention, but only with explanatory intention.

The *empirical-inductive proof schemes* could be related to the meaning of mathematical proof in scientific domains, which students share in their sciences classes. These schemes are based on the subjective conviction given by the verification of a proposition, in various particular cases.

These experimental features, originally coming from scientific contexts, are included in the mathematical proof. They induce ways of argumentation and personal schemes in mathematical proof which lack the validating power of deductive proof, but which serve to provide certain logical consistence to the conjectures obtained by intuitive procedures.

The *informal deductive schemes* could be related to the not very elaborated forms of mathematical proofs that mathematics teachers often use in the classroom; they are argumentations with a strong intuitive component, including visualisation (for example, proofs in differential calculus based on graphic representations of functions).

The students' *formal deductive proof schemes* could be related to the usual ways that mathematicians and mathematics teachers use to prove in a more rigorous way, using some type of formalism.

5. CONCLUSIONS

As a result of this study we have identified a variety of mathematical proof schemes in students who start their careers at the University of Córdoba (Spain) and we have related these proof schemes to different meanings of mathematical proofs in different institutional contexts. An important result obtained in this study is the very limited ability of these students to spontaneously produce deductive mathematical proofs even for elementary propositions.

It is necessary to somehow link the different meanings of proof, at different teaching levels, thereby progressively developing, among the students, the knowledge, the discriminative capacity and rationality required to apply them in each case. Informal proof schemes should not be considered as simply incorrect, mistaken or deficient, but rather as facets of mathematical reasoning necessary to achieve and master mathematical argumentative practices. The analytical arguments, which are characteristic of mathematical proofs, are not the sole argumentation practices used by mathematicians to convince themselves about the truth of their conjectures. These reasoning procedures are often unfruitful, or even an obstacle, in the creative/discovery stages of problem solving processes, in which it is allowed and even necessary to implement substantial ways of argumentation, in particular, empirical induction and analogy. We might recall Polyá's words (1944, p. 116): "Mathematics presented with rigor is a systematic and deductive science, but mathematics in gestation is an empirical and inductive science".

Understanding and mastering deductive argumentation by students requires a development of rationality and a specific state of knowledge. It demands "the adhesion to a problem that it is not that of the efficiency (exigency of practice) but rather that of rigor (theoretical exigency)" (Balacheff, 1987, p. 170). But the construction of this rationality is a progressive process that takes time, as well as ecological adaptations of the 'proof object' at different levels of teaching.

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