

# ONTO-SEMIOTIC CONFIGURATIONS UNDERLYING DIAGRAMMATIC REASONING<sup>1</sup>

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*Diagrams and in general the use of visualization and manipulative material play an important role in mathematics teaching and learning processes. Although several authors warn that mathematics objects should be distinguished from their possible material representations, the relations between these objects are still conflictive. In this paper, some theoretical tools from the onto-semiotic approach of mathematics knowledge are applied to analyse the diversity of objects and processes involved in mathematics activity, which is carried out using diagrammatic representations. This enables us to appreciate the synergic relations between ostensive and non-ostensive objects overlapping in mathematical practices. The onto-semiotic analysis is contextualised in a visual proof of the Pythagorean theorem.*

## INTRODUCTION

The use of different representations, visualizations, diagrams, manipulative materials, are proposed to favour mathematics learning by assuming that such materials make up representations of mathematics concepts and of the structures in which they are organised. It is supposed that the use of material representations is necessary, not only to communicate mathematical ideas but also for their own construction. However, the relations between representations, objects and construction of meanings are still conflictive. This issue is key for mathematics education since “any didactic theory, at one moment or another (unless it voluntarily wants to confine itself to a kind of naïve position), must clarify its ontological and epistemological position” (Radford, 2008, p. 221).

Researches in diagrammatic reasoning and about the use of visualizations in mathematics education do not usually deal with the type and diversity of mathematical objects. In this paper, this problem is faced using some theoretical tools from the onto-semiotic approach (OSA) (Godino, Batanero, & Font, 2007; Font, Godino, & Gallardo, 2013). Mathematical objects are considered to be abstracts whereas diagrams are specific and perceptible. It is necessary not confuse them, but the relationship between both types of objects are not dealt with explicitly. This situation is not strange

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since to clarify what abstract objects are, and their relationship with the empirical world, is a full-scale philosophical and psychological problem, which is addressed from different paradigms and theoretical frameworks.

In the OSA it is assumed that mathematics is a human activity (anthropological postulate) and that the entities involved in this activity come or emerge from the actions and discourse through which they are expressed and communicated (semiotic postulate). The epistemological, semiotic, and educational problem that interests us is to clarify the relationship between the visual, diagrammatic or iconic representations, and the non-ostensive mathematical objects that necessarily are involved.

In the following section, some characteristic features of the diagrammatic reasoning that point out the problem mentioned are described, that is the gap between the representation and the mathematical object represented. Then, the notion of ontosemiotic configuration of practices, objects and processes is summarised. This theoretical tool is used to analyse the diagrammatic reasoning in a visual proof of the Pythagorean theorem. In the final section, some reflections about the type of understanding that the onto-semiotic approach to mathematical knowledge might provide to diagrammatic reasoning are included.

## **DIAGRAMMATIC REASONING**

In mathematics education, talking of diagrammatic reasoning means entering into the field of Peircean Semiotics (Dörfler, 2005; Bakker & Hoffmann, 2005; Rivera, 2011), although the use of diagrams as a resource of thought and scientific work is also found in other fields and disciplines (Shin & Lemon, 2008).

A double conception about the notion of diagram is found: one wider conception, in which any type of inscription that makes use of the spatial positioning in two or three dimensions (right, left, forward, backward, etc.) is a diagram (geometric figures, graphs, conceptual, etc.). Another more restricted conception requires being able to carry out specific transformations, combinations or constructions with these representations, according to certain specific syntactic and semantic rules. In this research report, it is justified why this second approach should be retained.

Diagrammatic reasoning involves three steps (Bakker & Hoffmann, 2005, p. 340): the first step is to construct a diagram (or diagrams) by means of a representational system; the second step is to experiment with the diagram (or diagrams); the third step is to observe the results of experimenting and reflecting on them.

Duval (2006) attributes an essential role not only to the use of different systems of semiotic representation (SSR) for mathematics work but also to the treatment of the signs within each system and the conversion between different SSR:

The role that signs play in mathematics is not to be substituted for objects but for other signs! What matters is not representations but their transformation. Unlike the other areas of scientific knowledge, signs and semiotic representation transformation are at the heart of mathematical activity. (Duval, 2006, p. 107)

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Dörfler (2005) recognises that diagrams can make up a register of autonomous representation to represent and produce mathematics knowledge in certain specific fields; however, it is not complete. It requires to be complemented by conceptual-verbal language in order to express notions like: continuity and differentiability; impossibility that specific objects exist; using the quantifiers ‘for all’, ‘each one’ and ‘there are’.

For our purposes here, it is very important to make a clear distinction between "diagrams" and all kinds of representations, visualizations, drawings, graphs, sketches, and illustrations as widely used in professional mathematics and in mathematics education as well. Although these might be diagrams in the specific sense used here, this is mostly not the case. This is due to the lack of the constituting operations by which an inscription or visualization becomes only a diagram. (Dörfler, 2005, p. 58)

Shin & Lemon point out another problem related to the use of diagrams:

A central issue, if not the central issue, was the generality problem. The diagram that appears with a Euclidean proof provides a single instantiation of the type of geometric configurations the proof is about. Yet properties seen to hold in the diagram are taken to hold of all the configurations of the given type. What justifies this jump from the particular to the general? (2008, section 4.1)

Sherry (2009) adopts an anthropological perspective on the role of diagrams in mathematics argumentation, which involves an objectification of the empirical reality. This perspective differs from the Peircean semiotic, according to which diagrams are an essential means in the process of hypostatic abstraction. Sherry analyses the role of diagrams in mathematics reasoning (geometric and numerical – algebraic) without resorting to the introduction of abstract objects and relying on a Wittgensteinian perspective of mathematics. “Recognizing that a diagram is just one among other physical objects is the crucial step in understanding the role of diagrams in mathematical argument” (Sherry, 2009, p. 65).

In this position, the author avoids recurring to abstract conceptions which are conceived in an empirical-realistic way (hypostatic abstraction) in order to understand them as socially agreed grammatical rules, about the use of languages through which we describe our worlds (material or immaterial).

I have emphasized that diagrammatic reasoning recapitulates habits of applied mathematical reasoning. On this view, diagrams are not representations of abstract objects, but simply physical objects, which are sometimes used to represent other physical objects. (Sherry, 2009, p. 67)

## **ONTO-SEMIOTIC CONFIGURATIONS**

In the OSA framework, it is proposed that six types of objects intervene in mathematics practice, which can be contemplated from five dual points of view (figure 1) (Font et al., 2013). The non-ostensive (immaterial) entities: conceptual, propositional and procedural, are conceived as rules. The Wittgenstein’s anthropological view is assumed, according to which concepts, propositions and mathematics procedures are

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empirical propositions, which have been socially reified as rules. Sherry clearly and synthetically describes this Wittgensteinian conception of mathematical objects:

In order for an empirical proposition is harden into a rule, there must be overwhelming agreement among people, not only in their observations, but also in their reactions to them. This agreement reflects, presumably, biological and anthropological facts about human beings. An empirical proposition that has hardened into a rule very likely has practical value, underwriting inferences in commerce, architecture, etc. (Sherry, 2009, p 66)

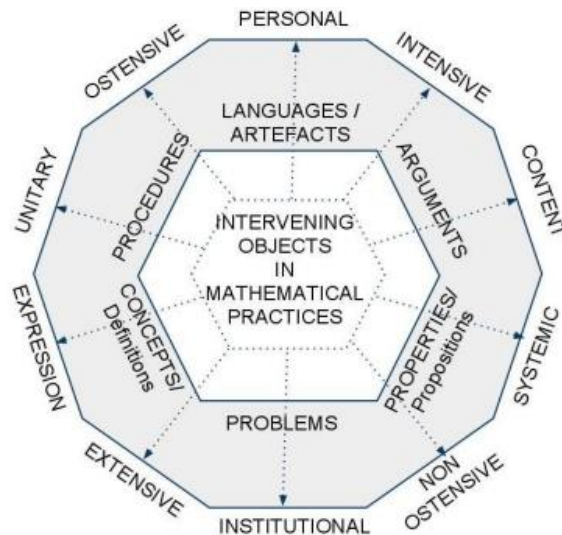


Figure 1: Objects that intervene in mathematical practices (Font et al., 2013, p. 117)

Both the dualities and the configurations of primary objects may be analyzed from the process/product perspective. The objects of a configuration (problems, definitions, propositions, procedures and arguments) emerge through the respective mathematical processes of communication, problematization, definition, enunciation, development of procedures (algorithms, routines, etc.) and argumentation. For their part, the dualities give rise to the following cognitive/epistemic processes: institutionalization-personalization; generalization-particularization; analysis / decomposition - synthesis / reification; materialization / concretion - idealization / abstraction; expression / representation - signification.

Behind diagrammatic reasoning, and the use of manipulative teaching materials, there is an implicit adoption of an empirical – realistic position about the nature of mathematics. This position does not recognize the essential role of language and the social interaction in the emergence of mathematical objects. To a certain extent, it is supposed that the mathematical object “is seen”, it is hypostatically detached from empirical qualities of things collections. Against this position, the anthropological conception of mathematics proposes that concepts and mathematical propositions should be understood, not as hypostatic abstractions of perceptual quality, but as regulations of the operative and discursive practices carried out by people in order to describe and act in the social and empirical world in which we live.

This anthropological way of understanding abstraction, that is, the emergence of general and immaterial objects forming mathematical structures, has important consequences for mathematics education since mathematics learning should take place through students' progressive participation in the mathematics language games. For example, in the current introduction of dynamic software in school is necessary to evolve their use according moments of exploration, illustration and demonstration (Lasa & Wilhelmi, 2013), which allow an understanding, reuse and construction of new mathematical knowledge. In this way, dialogue and social interaction take on an important role, in comparison with the mere manipulation and visualization of ostensive objects.

### ONTO-SEMIOTIC CONFIGURATION IN A VISUAL TASK

In this section, the types of practices, objects and processes put at stake in the statement and demonstration of the Pythagorean theorem are analysed. Usually it is presented as a visual or "without words" demonstration. It is shown that, indeed: "picture-proofs don't show their results on their sleeve, as it were; it's necessary to study them for a while, before they reveal their treasure" (Sherry, 2009, p. 68).

#### Task

*What is the relationship between the areas of the figures shaded A and B?*

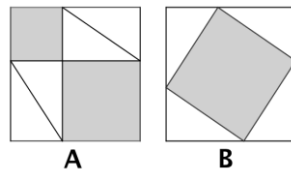


Figure 2: A visual proof of the Pythagorean theorem

The following sequence of operative and discursive practices is one possible answer<sup>2</sup>:

1. We assume that the representations in Figure 2 are squares and right triangle, and the lengths of their sides are indeterminate:  $a$ ,  $b$ ,  $c$  (Figure 3).
2. The quadrilaterals formed by the outer segments of the figures A and B are congruent squares because the sides have equal length,  $(a + b)$ .

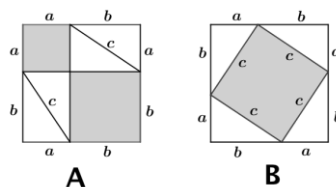


Figure 3: Metrics hypothesis needed

3. The representations of right triangles in A and B are congruent because their sides are of equal length.

<sup>2</sup> Explanatory proof (Cellucci, 2008).

4. The shaded region in Figure A is equal to the shaded region in Figure B. This is because two squares of equal area are formed of four equal triangles.
5. The shaded area in Figure A is the sum of the squares area of sides  $a$  and  $b$ , respectively,  $a^2 + b^2$ .
6. The shaded area in Figure B is the square's area of side  $c$ ,  $c^2$ .
7. The shaded regions are interpreted as areas of the squares whose sides are the legs and hypotenuse of the triangle, respectively (Figure 4).

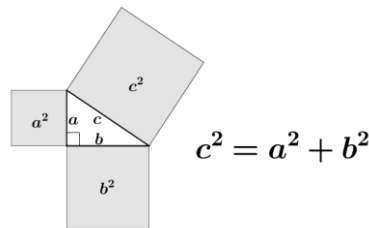


Figure 4: Determination of the Pythagorean theorem

8. Then, the area of the square on the hypotenuse is equal to the sum of the squares areas on the other two sides:  $c^2 = a^2 + b^2$ .

### Configuration of objects and meanings

In the first column of the Table 1, the expressions in ordinary language (sequential) is summarised; such expressions are added to the diagrams to produce the justification and explanation necessary of the theorem. In the second column, the system of ‘non-ostensive objects’ is included. In addition, how the ‘ostensive / non-ostensive’ duality, and the “example / type” (particular / general) duality are linked to the intervention of concepts, propositions, procedures, and arguments are shown.

OSTENSIVE OBJECTS	NON - OSTENSIVE OBJECTS
(Means of expressions)	(Concepts, propositions, procedures, arguments)
Task statement: <i>What is the relationship between the areas of the figures shaded A and B?</i> (Figure 2)	<i>Concepts:</i> area (extension of a plane region), sum of areas; comparison of areas. <i>Particularization:</i> these concepts are particularized to the case of the figures given. The squares, triangles and the relationships between the areas, are generic.
1. We assume that the representations in Figure 2 are squares and right triangle, and the lengths of their sides are indeterminate: $a, b, c$ (Figure 3).	<i>Concepts:</i> square, right triangle, side, indeterminate measurement of length. <i>Particularization:</i> these concepts are particularized to the case of the figures

	given.
	The figures refer to square and triangle generics. The lengths are generic.
2. The quadrilaterals formed by the outer segments of the figures A and B are congruent squares because the sides are of equal length, $(a + b)$ .	<p><i>Proposition:</i> the two exterior squares are congruent.</p> <p><i>Argumentation:</i> because the sides of the squares have the same length. This is <math>(a+b)</math>.</p> <p>The proposition is general; it is valid for the “examples” (figures) and for any “type”. This is an essential hypothesis in the explanatory process.</p>
...	
8. Then, the square's area of the hypotenuse is equal to the sum of the squares areas of the other two sides: $c^2 = a^2 + b^2$ .	<p><i>Proposition:</i> thesis (Pythagorean theorem)</p> <p><i>Justification:</i> steps 1 to 7. It is geometrically interpreted (comparison of areas). It is also interpreted in arithmetic / algebraic terms (numerical relationships).</p>

Table 1: Configuration of objects and meanings

Our analysis agrees with and supports Sherry’s position about the use of diagrams in mathematics work: rather than building an accurate diagram, what matters is the mathematical knowledge involved, which is not visible anywhere; it is not in the diagrams themselves. In the case of using dynamic software, it is essential to progress from moments of illustration (where objects can be manipulated with great precision) to moments of demonstration (where objects are not essential, rather the construction process of diagrams). This way, features of specific *examples* can progress towards the corresponding structural *type*. In general, the diagram supports or makes possible the necessary process of particularization of the general rule; it makes the conceptual object intervene in order to participate in a practice from which another new conceptual object will emerge (in our example, Pythagorean theorem).

## FINAL CONSIDERATIONS

The function that we attribute to the diagrams helps to surpass ingenuous empiricist positions about the use of manipulatives and visualizations in the processes of mathematics teaching and learning: there is always a cohort of intervening non material objects which are essential to solve these situations accompanying the necessary materializations that intervene in the situations-problems and the corresponding mathematics practices. However, this layer of material objects should not prevent seeing the layer of immaterial objects that really make up the conceptual system of institutional mathematics. Both layers are interwoven and to a certain extent

are inseparable. Mathematics teacher should have knowledge, understanding and competence in order to discriminate the different types of objects that intervene in school mathematics practice, based on the use of different systems of representations and being aware of the synergic relations between the same.

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