The proposal we present is part of a broader research, and it is based on didactic experiences with in-service teachers, in which we confirm that developing problem posing tasks that facilitate or delve into learning reveals deficiencies in teachers’ competence in didactic analysis. Using epistemic and cognitive configurations, which are typical tools of the onto-semiotic approach of cognition and mathematics instruction (OSA), we propose to stimulate in-service teachers’ competence in didactic analysis using a problem posing strategy with a phase to reflect on the mathematical practices carried out and the objects associated to it.

INTRODUCTION

Problem posing is a productive field to exercise teaching competences (Crespo, 2003; Malaspina, Mallart & Font, 2015). In this research, we emphasize on this strategy’s connection to competences in didactic analysis, which are exposed when problems are posed in order to facilitate the understanding and solution of a problem proposed in a classroom environment (episode).

We consider that a math teacher’s key professional competence is the one in didactic analysis of instruction processes (Giménez, Font & Vanegas, 2013; Rubio, 2012). The importance reflected by the didactic analysis is related to the possibility a math teacher will have to design, implement, assess and improve his teaching work. In order to examine and stimulate this competence, we will take into consideration some theoretical constructs of the onto-semiotic approach of cognition and mathematics instruction (OSA) (Godino, Batanero, & Font, 2007), especially at the analysis level of mathematical objects activated when developing a mathematical practice in tasks involving problem posing. Thus, we adopt the lines of research that use problem posing as a window of opportunities for students and teachers to understand mathematics, as well as the studies of specific strategies for problem posing (Kontorovich & Koichu, 2009).

In this study, we consider the following research question: how can we use problem posing to stimulate the development of teachers’ competence in didactic analysis? To answer this question, we propose a problem posing strategy which includes OSA tools, such as epistemic and cognitive configurations, from problem posing experiences developed in workshops with in-service teachers.

THEORETICAL FRAME

Problem posing and competences of a mathematics teacher

We assume that problem posing is a process through which a new mathematical problem is obtained by varying a given problem or by elaborating a problem, whether facing a specific situation or due to a specific request of mathematical or didactic nature (Malaspina, 2015).

In this research, we will focus on problem posing by varying a given problem, emphasizing the didactic perspective of such problems. This didactic aspect should facilitate the understanding and
solution of a problem proposed in a classroom environment (episode), and it will be used to analyze the teacher’s mathematical practice.

In order to develop and assess students’ competences, the assigned tasks are focused on problem solving. In that sense, the teacher must not only have the ability to solve mathematical problems, but also to choose, modify and pose problems with educational purposes (Malaspina, 2015). This process, where the mathematics teacher analyzes the problem for educational purposes, implies the development of a competence in didactic analysis, which involves reflecting on his mathematical practice of problem solving and posing, assessing problems based on certain didactic criteria.

We agree with Rubio (2012), since we consider that competence in didactic analysis of instruction processes refers to designing, applying and assessing one’s learning sequences as well as others’, using didactic analysis techniques and quality criteria in order to set cycles to plan, implement, assess and improve those processes. Regarding teacher training, this competence has to be developed with tasks that involve managing the didactic analysis technique (Malaspina, 2015). One of these tasks consists of posing problems and reflecting on them didactically.

**Onto-semiotic approach of cognition and mathematics instruction (OSA)**

The Didactic-Mathematical Knowledge (DMK) model is proposed in the OSA in order to analyze teachers’ mathematical and didactical knowledge (Godino & Pino-Fan, 2013). From a global perspective, the DMK model consists of a tool called didactic analysis, which can be used by the same teachers to look into their own practice in six facets or dimensions: epistemic, cognitive, affective, mediational, ecological and interactional. In this research, we focus on the epistemic and cognitive dimensions. For each one of these facets proposed in the DMK model, five levels of didactic analysis are considered in turn: 1) analysis of problem types and practice systems; 2) configuration of mathematical objects and processes; 3) analysis of didactic paths and interactions; 4) identification of the system of norms and meta-norms; and 5) assessment of the didactic suitability of the instruction process. In this investigation, we will limit ourselves to the first two levels of didactic analysis, without considering the mathematical processes, since they imply a greater complexity that escapes the purposes of this work. Therefore, we pretend to study the objects that emerge and intervene as a consequence of the problem posing process, using the *epistemic configuration* (EC) and *cognitive configuration* (CC) tools, which we explain and exemplify later on (see Figure 1).

According to Godino, Batanero and Font (2007), when a person carries out a mathematical practice and assesses it, he/she has to activate a mixture formed by some or all of the mathematical objects, that is to say: problem situations, languages, propositions, definitions, procedures and arguments. These objects will be interrelated, making configurations defined as webs of objects that intervene and emerge from the systems of practice; such configurations are epistemic (EC) when they are webs of objects considered from an institutional perspective, and they are cognitive (CC) when they are webs of objects considered from a personal perspective. Analyzing these configurations allows us to obtain information about the *anatomy of a problem*.

We will use these OSA tools to study the mathematical objects that intervene in problem posing as a product of a mathematical activity. Additionally, we propose to explain the four basic elements of every problem in the problem situations of the EC and CC, according to Malaspina (2015):
information, requirement, context (intra-mathematical or extra-mathematical) and mathematical environment. As we will see later, that is how we do it in this research, and it is especially relevant, taking into account that these elements served as reference to pose problems with an emphasis on the didactic point of view.

Figure 1. Configuration of primary objects

**METHODOLOGY**

In this research we used a case study and analyzed two types of data: problems posed by in-service teachers as well as EC and CC based on solutions elaborated by experts. The data for this study was taken from 15 in-service high school mathematics teachers who participated in the *Workshop on Posing Problems with Functions*. When we analyzed the posed problems, we used the analysis of the content to classify them according to the implicit intention of the activity. For the qualitative study of the configurations as a result of the analysis of solutions for episode’s problem and the posed problems, we used the technique called onto-semiotic analysis (Godino, 2002), which allows describing systematically the mathematical activity carried out by the in-service teachers (mathematical practice of resolution), the mathematical activity of posing (mathematical practice of posing), and the primary mathematical objects (language, problem situation, concepts, propositions, procedures and arguments).

In the *Problem posing workshop*, we used the strategy to stimulate the development of the ability to pose problems by varying a given problem presented by Malaspina, Mallart and Font (2015), which considers an *episode* in class, posing a *pre-problem* and a *post-problem*. To make this easier, we will name this strategy EPP. The *episode* in class can be summarized as the request to solve a problem and some students’ reactions while thinking to solve it; the pre-problem and post-problem have to be posed by the teachers participating in the workshop. The main feature of the pre-problem is that understanding and solving it help understand and solve the problem posed in the episode; and the main feature of the post-problem is that it be inspired by the problem of the episode, and that it be more challenging. We consider 4 phases in this research: (1) design a class episode, as well as problem solving and problem posing tasks to propose them to the teachers, which includes elaborating an EC based on an expert solution to the problem of the episode; (2) apply the tool elaborated according to the EPP strategy; (3) analyze the problems solved and posed by the in-
service teachers using the elaborated EC and CC; and (4) elaborate the proposal based on the analysis and comparisons of the elaborated EC and CC. Following is the episode presented to the participants:

Mr. Torres proposed the following problem to eighth-grade students in a math class on functions:

In the shop at the corner of my block, each kilogram of potatoes costs S/.3. In the Wholesale Market, which is far from home, each kilogram of potatoes costs S/.2, but I have to spend S/.5 in bus tickets to get there and come back. Will it always be convenient to buy potatoes at the Wholesale Market? Why?

After a few minutes, some students commented:

Juan: Sure, it will always be more convenient to shop at the market because it is cheaper there.

María: Not always… It depends…

Mateo: It will be more convenient to shop at the market if you have to buy more than 8 kilos of potatoes.

An expert solution was adopted for the problem, and the EC of such solution was made in order to have it as a reference to analyze and compare it to the CC of the participants’ solutions (CCPp).

**Expert solution and EC of the episode problem (ECPe)**

The expert solution to this problem is based mainly on the definition of two functions: \( f(x) = 3x \) and \( g(x) = 2x + 5 \), which express how much \( x \) kilos of potatoes cost in a convenience store and in the wholesale market, respectively. The graphs for both functions are drawn in the same system of coordinates, and it is determined – visually and algebraically – that the amount spent in \( x \) kilos of potatoes is not always less in the wholesale market than in the convenience store.

While elaborating the EC of the solution, the languages used (verbal, symbolic and graphic representations), the information, requirement, context and mathematical environment of the problem are explicitly stated; the concepts involved (linear function, affine function, expense function, slope, \( y \)-intercepts, graphs of functions, linear inequation) and the emerging proposition: If there are values of \( x \) for which \( f(x) < g(x) \), then it is not always more convenient to buy in the wholesale market. In addition, the procedure is described and the arguments explained to tell the truth about the given proposition, which derives in the conclusion.

**RESULTS**

In relation to the solutions of the problem shown in the episode above, most of them do not reveal similar procedures to the ones of the expert solution. The use of tables prevails, as well as the calculations of the expenses in specific cases. Eight teachers define the functions of the case, but only 2 use their graphs. On the other hand, only 2 solve an inequation and only 4 use the expression not always in their answer. A thorough analysis of the CC of the participants’ solutions reveals important aspects of their mathematical competence, which we will not explain in detail now. In relation to the problems posed by the teachers, in general terms, they show the ability to pose problems by varying a given problem; however, we perceive little consideration to the didactic characteristic such problems should have, like Pre-problem, in the sense of helping students to clarify and solve the given problem. To make this perception more evident, we elaborated CCs from the solutions that teachers presented, requested expert solutions to the proposed problems, elaborated ECs from such solutions, and did some comparisons between these configurations, which revealed important aspects of the teachers’ didactic competences. Some of them gave us
information related strongly to mathematical competences (M), and others gave us information related strongly to the didactic analysis (D). In Figure 2, a diagram to compare configurations is shown.

![Diagram to compare configurations](image)

**Figure 2. Diagram to compare configurations**

As an example of the pre-problem posing task, in Figure 3 we present a problem posed by an in-service teacher (we call him P3).

![Pre-problem posed by P3](image)

**Figure 3. Pre-problem posed by P3**

In Figure 4, we show the teacher’s solution to his posed problem:

![Teacher P3’s solution to his posed problem](image)

**Figure 4. Teacher P3’s solution to his posed problem.**

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**Translation:**

Lalo is a boy who makes a living selling roses, earning S/. 2 each. To make sure he sells out, he decides to go to a concert of romantic music, where he has to pay S/. 20 to get in. How many roses will he have to sell to beat his regular sale, with which he earns S/. 30 daily in average?

Let \( f(x) = \text{profit} \)

\[ x = \text{amount of roses} \]

1) To break even

\[ \text{Answer: } 26 \text{ roses minimum} \]

2) To earn S/.30

---

**Profit**  
**Concert ticket**
According to the methodology presented, we elaborated the CC of this solution (CCPp), which will be compared to the EC of the episode problem (ECPe) summarized above.

Table 1. Cognitive configuration of the solution to the posed pre-problem (CCPp)

<table>
<thead>
<tr>
<th>LANGUAGES</th>
<th>PROBLEM SITUATION</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Verbal representations:</strong></td>
<td><strong>Information:</strong> Income for selling a</td>
</tr>
<tr>
<td>Profit, ticket, minimum,</td>
<td>product, considering a fixed cost.</td>
</tr>
<tr>
<td>amount, loss, average, sales,</td>
<td>Daily average income specified.</td>
</tr>
<tr>
<td>variable ( x ), amount of</td>
<td>Requirement: Determine the minimum</td>
</tr>
<tr>
<td>roses sold, function ( f ):</td>
<td>amount of units that need to be sold</td>
</tr>
<tr>
<td>profit</td>
<td>to beat a given income amount.</td>
</tr>
<tr>
<td><strong>Symbolic representations:</strong></td>
<td><strong>Context:</strong> Extra-mathematical</td>
</tr>
<tr>
<td>Amount of roses: 1, 2,</td>
<td>Mathematical Environment: Related</td>
</tr>
<tr>
<td>3,…, 13, 28. Profit: 2, 4,…,</td>
<td>affine functions, linear equations.</td>
</tr>
<tr>
<td>( f(x) = 2x - 20 ), Soles</td>
<td></td>
</tr>
<tr>
<td>(S/), =</td>
<td></td>
</tr>
<tr>
<td><strong>Numerical representations:</strong></td>
<td></td>
</tr>
<tr>
<td>Table with organized</td>
<td>CONCEPTS</td>
</tr>
<tr>
<td>information about the amount</td>
<td>Linear function, graphs of functions,</td>
</tr>
<tr>
<td>of roses and its corresponding</td>
<td>( x )-intercepts, strictly</td>
</tr>
<tr>
<td>profit, as well as a column</td>
<td>increasing function, income</td>
</tr>
<tr>
<td>for the cost of the concert</td>
<td>function, profit function, linear</td>
</tr>
<tr>
<td>ticket.</td>
<td>equation, variable, algebraic expression, minimum whole number of a lower enclosed set of real numbers.</td>
</tr>
</tbody>
</table>

![Table](table.png)

**PROPOSITIONS**

The solution to the equation \( f(x) = k \) determines the number of units that need to be sold to have a profit \( k \).

If \( f(x) = 0 \), there are no earnings or losses.

**PROCEDURES**

Define variable \( x \). Write algebraically the profit function \( f \) for selling \( x \) roses, taking only in consideration a fixed cost (payment for the concert ticket). Elaborate a table to place the resulting amounts from substituting variable \( x \) for positive whole numbers. Compare the results in the table, taking into account the number obtained as income. Determine the number of units required to sell in order to get the given profit, using the determined function. Graph the profit function and observe that it is strictly increasing. Conclude that the minimum number required will be the lowest whole number greater than the number gotten in the previous step.
ARGUMENTS

**Thesis 1:**
If \( f(x)=0 \), there are no earnings or losses.

**Argument:**
A positive profit is a “real profit” and a negative profit is a loss.

**Thesis 2:**
If \( f \) is a strictly increasing function, \( q \) is a given number and \( u \) solves equation \( f(x) = q \), then the lowest whole number \( v \), greater than \( u \), is the lowest whole number, as long as \( f(x) > q \) for every \( x \geq v \).

**Argument:**
Since function \( f \) is strictly increasing, if \( v > u \), then \( f(v) > f(u) = q \). Since \( v \) is the lowest whole number greater than \( u \), any \( x \) number, specially a whole number, as long as \( x \geq v > u \) complies with \( f(x) > f(u) = q \). That is, \( f(x) > q \).

Analyzing the CC of the solution to this posed problem (Table 1), we state the problem has some good characteristics such as a clear, interesting statement with an extra-mathematical context and related to the mathematical environment desired to work with eighth-grade students. Nevertheless, it is evident that in this problem posed by P3 and in the solution he proposes himself, the concepts, propositions, procedures and arguments – even though these last ones are not explicit in his solution – require a greater cognitive demand than the problem of the episode. We conclude that the problem posed does not have the conditions of a Pre-problem, whose main characteristic is to facilitate the understanding and solution of the problem given in the episode. In general terms, we could say that the posed and solved problem reveals mathematical competence and absence of didactic analysis. This happens in every case of the Pre-problem posed; for this reason, we consider that it is necessary to polish the EPP strategy, including a stage of reflection on the mathematical practice of problem solving and problem posing, which could serve as the basis for teachers to improve their didactic analysis in relation to problem posing.

**FINAL CONSIDERATIONS**

In virtue of the analyses performed and the importance of developing teachers’ competence in didactic analysis, particularly when posing math problems that contribute to facilitate and carry on into their students’ learning, we propose the EPP strategy for problem posing to be polished, considering an (R) phase of metacognitive and didactic reflection; therefore, the strategy name would be ERPP. In such phase, the teacher must elaborate a CC of his/her solution to the problem of the episode and – based on it – reflect on his/her practice through a brief narration (addressed to a colleague) – of the fundamental steps he/she took in order to solve the problem. Also, once the requested Pre-problem is posed and solved, he should elaborate a CC of such solution, and then make a supported comment regarding how convinced he/she is that the problem posed will contribute to the right understanding and solution of the problem of the episode. Something similar should be done with the Post-problem, but this one should be more challenging. All of this certainly requires a previous session with the teachers, where at least the EC of the solution of the problem is elaborated jointly, considering a mathematical environment related to the environment desired to work in class.
Acknowledgement

This research was carried out as part of two projects: EDU2015-64646-P, Ministry of Economy and Competitiveness (MINECO, Spain) and IREM-PUCP 0390 (Peru).

References


