FROM LINEAR ALGEBRA TO GEOMETRIES: DIDACTIC PROPOSAL BASED ON THE ONTOSEMIOTIC APPROACH

L. Sbitneva[1], N. Moreno[2], R. Valdez[1]

[1] Universidad Autónoma del Estado de Morelos (MEXICO)
[2] CINVESTAV (MEXICO)

Abstract

The main objective of our educational project is to provide the undergraduate students in the modality of Blended Learning with the didactic materials as a tool for friendly introduction to geometric interpretations of mathematical models which involves linear equations, so that students, without previous fundamental knowledge, could be able to develop gradually the geometric intuition and visualization. of 2 and 3 dimensional geometry.

The notion of semiotic function and the unitary/systemic duality coming from the ontosemiotic approach (OSA) is adopted to explain the meaning associated to the mathematical objects and their complexity.

The viability of the proposal was verified through the analysis of responses to a questionnaire applied to our students.

Keywords: Linear Algebra, Geometry, Semiotic function, Ontosemiotic Approach.

1 INTRODUCTION

The majority of the undergraduate mathematics courses deals with the models which require coordinate description of the mathematical objects of different nature useful in applications in other branches of sciences, which supposes the familiarity with the basic concepts of numbers as well as relative concepts of geometric nature, in order to give a graphic representation of phenomena of linear algebra limited to two and tree dimensions, because the nature has provided us with a ready-made "demarcation line" in endowing us with geometric intuition for spaces of these dimensions.

Taking into account the unitary/systemic perspective [1], this phenomenon can be seen in the scholar context when in university courses some mathematical objects can be considered as something known as a unitary entity, meanwhile in high school, those are considered as complex (different representations, properties, problems in applications, etc.).

Nowadays, coordinates appear as a result of the axiomatic method in teaching Linear Algebra courses, nevertheless almost all learning books suppose the previous knowledge on geometry, which is not the case due to educational innovations during last decades. Mathematicians have been long aware of such phenomena where the replacement of one axiomatic system by another equivalent but more suitable system, sometimes leads to considerable simplifications. However, this does not always produce the desirable effect on student's learning. Considering our experience at various Universities in Mexico, we have seen that many students are not capable to relate proper meaning to many mathematical objects in analytic geometry and solve problems operatively, nevertheless this situation can be improved in rather short time while the discursive practice on the historical development of geometries in past centuries..

This educational project is based on our experience to teach analytic geometry using the advantages of linear algebra because it allows all the development of "elementary geometry" to be set forth quite impeccably and even effortlessly" according to the Jean Dieudonné ´proposal (Dieudonné, J. 1969).

The sequence of the didactic materials for b-learning modality starts from a system of linear equations on two and three variables so that the so called "Analytic geometry" is developed rather naturally. Our main objective is to introduce the fundamental geometric concepts like straight line, planes and interrelation between them, emphasizing different types of geometric properties.

The methodology for implementation of our innovations is based on some theoretical and methodological elements developed in the frames of the Ontosemiotic (OSA) approach [1], [2].

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In OSA approach the mathematical activity plays a central role and is modeled in terms of systems of operative and discursive practices. From these practices, the different types of related mathematical objects (language, arguments, concepts, propositions, procedures, and problems) emerge, building cognitive or epistemic configurations among them.

1.1 Geometry and Linear Algebra

Geometry is the fundamental part of Contemporary Mathematics and of vast field of applications.

In contrast, the education of Geometry at school level is gradually decreased. Even at the University level, the classic course of Analytic geometry is almost absent, making an auxiliary part of the Calculus courses.

The courses of Euclidean geometry, Greek geometry, which were obligatory at secondary education, now is reduced to a collection of fragments concerning some geometric figures.

There were some reasons to these changes due to processes of formalization in the Mathematics itself: the emphasis was put on the strict foundations of Set Theory,

Nowadays, once more the geometry recovers its leading position due to the development of hyperbolic and projective geometries useful in applications in modern physics and computer sciences. This new vision of geometry was formulated by F. Klein in the Erlangen Program:

Geometry is the study of the spaces with its transformations.

This approach is best treated in the terms of the apparatus of Linear Algebra, permitting the description of three fundamental geometries: Euclidean and non-Euclidean, hyperbolic and elliptic, the latter two are as valid as the Greek geometry discovered experimentally since ancient time.

Nowadays the Linear Algebra course is the only basic mathematics course for many specialties,

We should emphasize that for the scientific formation of students for mathematics and physics, this course was usually preceded by the course of analytic geometry, this is why almost all learning books make the references to the traditional knowledge of geometry of 2 and 3 dimensions, including operations with vectors.

1.2 The Ontosemiotic Approach to the Mathematical Objects

Theoretical elements from the Ontosemiotic Approach, OSA [2,3], have been employed to describe a way in which the meaning of the mathematical objects can be achieved. In OSA, mathematical activity plays a central role and is modeled in terms of systems of operative and discursive practices in which the meaning of a mathematical object involved in a practice is understood in two ways taking into account the facets expression/content and unitary/systemic [1]. Based on the expression/content perspective, the meaning of an object considered as “expression” in a semiotic function will be the "content" of the semiotic function established by someone following a rule or criterion of correspondence. From the perspective unitary/systemic the "meaning" of an object, depending on the context, can be a definition (unified approach) or it may be the system of practices in which the object is crucial for comprehension (systems perspective).

Through the notion of semiotic function we describe the importance of the establishment of relations between mathematical objects belonging to the geometric domain and to the linear algebra in order to establish the appropriate meanings and to help the students overcome the practice of memorizing mathematical procedures.

Each semiotic function involves an act of semiosis by an agent (a subject like a student or an institution) and constitutes a knowledge. Speaking of knowledge is equivalent to discuss how the subject establishes a semiotic function (or a pattern of such functions), resulting in a variety of types of knowledge in relation to a diversity of semiotic functions that can be established between the various entities introduced in the model.

2 CONCEPT OF COORDINATE SYSTEMS. DISCURSIVE PRACTICE

As it has been mentioned above, the concept of coordinate system appears in an axiomatic manner in teaching linear algebra courses. The learning books may present a discourse based on the following statements which need to be explained during discursive practice.

- A Cartesian system of coordinates in a plane may be given by the two *unit vectors* on its $+x_1$, $+x_2$ *axes*.
- Conversely, any two *mutually orthogonal unit vectors* attached to some point in the *plane* determine a system of Cartesian coordinates if we take them to define the *axes* with their *positive orientations***.**
- *The plane of Euclidean geometry* is quite distinct from a *two dimensional vector space.* In a vector space all parallel vectors of the *same length* and *same orientation* are identified.

  In Euclidean space the basic geometric object is a *vector together with its starting point*.
- The notion which will permit us to compute in the vector plane the phenomena of the Euclidean geometry is the notion of a frame. Further on, in applications, it is used to form a concept of a moving frame in the differential geometry.

The mathematical treatment based on these axioms generates a series of conflicts that can be observed in the classroom. A way to overcome these difficulties can be achieved through the establishment of a network of semiotic functions between objects belonging to Euclidean geometry and analytic geometry objects.

This work is not intended to give a detailed description of all the different semiotic functions to improve students understanding, rather we try to illustrate two relationships that are necessary to achieve a better understanding of geometry and algebra: first between geometries, Subsections 2.1 and 2.2, (Euclidian and analytic geometries) and then between analytic geometry and linear algebra, Section 3.

2.1 Cartesian coordinate system and Euclidean Geometry

The main problem is how to achieve the comprehension of different methods to introduce the relations between real numbers and the objects of geometric nature, in order to be able to make conclusions based on operations with these numbers, called coordinates, so that the correct characterization of geometric objects could be obtained (and vice-versa).

This can be seen from the systemic perspective, that is, we can say that a geometric object needs other mathematical objects from another mathematical domain for its constitution. For example: how one can decide that the triangle with the three vertices, named A, B, C, is a right-angle triangle, i.e. the Pythagoras Theorem holds; this suggests some complexity of mathematical objects that should not be underestimated.

Almost all school children should know that the Cartesian coordinates, assigned to each of this points, and the formula for the distance between any two points, applied to the three pairs of the vertices, gives three numbers a,b,c, which express the lengths of three edges. Comparing the squares of these numbers, one can verify if the sum of two squares is equal to the square of the third one; then, because of the inverse to the Pythagorean theorem, the triangle has a right angle.

The above arguments suggest to establish some set of relationships between certain physical intuitive knowledge (somehow considered in Euclidean geometry, like the notion of distance) and the domain of analytic geometry.

For example, although students usually remember the formula for the distance between two points, A, B, with the coordinates, say (x_1, y_1) and (x_2, y_2), they are not able to explain the meaning of these numbers assigned as the Cartesian coordinates to the points A, B marked on the blackboard. Usually, they show, making horizontal and vertical movements by their hands, that the cross represents the Cartesian coordinates.

Thus we start our discursive practice with these very gests, stating several questions-

- What does represent the horizontal movement?
- The horizontal line,
- but what is a line?, what does it mean horizontal?

No answer for the line, (once more only gests representation). But with the concept of Horizon we arrive to the non-ostensive object: since it is imaginary line which moves away as we get closer to it…

2.1.1 Perpendicular lines on a plane

Considering some empirical knowledge the discussion of the concept of verticality may have a more realistic hue: a constructive example is the thread with the piece of plumb attached at lower end, which produces the effect of the gravitational force.

But this does not work on the horizontal plane to draw a “cross” which states for the representative of Cartesian orthogonal system. The discussion of the concept of the perpendicularity and the corresponding construction in ancient time resulted in finding a practical device, the thread with 12 knots, which, being stretched with three vertices so that the sides of the triangle forms 3,4 and 5 segments equally spaced between knots, forms a triangle. Since $3^2 + 4^2 = 5^2$, it gives Pythagorean triangle, also named Egyptian triangle in architecture.

Thus the “cross” seems to be constructed. The students mark the point of intersection with O, the origin, and immediately put small vertical marks on the horizontal line and a series of small horizontal marks on the vertical line.

The question arises: Why there is only one point of intersection? What do represent the little marks?

If we can not give a definition of a straight line, nor for a point, how can we justify the construction?

The discussion leads us to the axioms of Euclidean geometry. We omit here this fruitful part of introduction, which clarifies the way of construction of the pair of unique points on both perpendicular lines, which are the orthogonal projections of a given point onto coordinate lines.

So far there are no numbers, which could be called coordinates: the next phase of the discussion is concerned to the construction of a numerical line, so called a coordinate axes.

2.1.2 Axes converting a geometric line to the numerical line:

There non possibility to treat this theme rigorously due to many reasons, thus we refer to this concept as is taught at the school level: the relative measure. Thus there appear the unit of measurement, marked on each line, on the right-hand side and above the origin (the discussion of the orientation is indispensable).

We proceed with an intuitive approach.

The straight geometric line, with an arbitrary fixed point on it, presents two possible directions (orientations), traditionally it is chosen to mark a point which will correspond to the number 1, as the end point of the unit of measurement placed along the line and attached at the origin.

There is no problem to mark the points which correspond to the natural numbers, and also so called the symmetric numbers on the left half of the line, which receive the name “negative numbers” (explications of the terminology due to historical development and relation to religion may be found in Internet)

Naturally the point of origin O receives the coordinate 0.

Construction of fractional numbers, between 0 and 1, requires the Euclidean geometry tools of drawing parallel lines.

First students record, or invent, the division of a segment into two equal parts, using the (rule and compass).

Division into three, 4, … k, etc parts requires the use of Theorem of Thales, in order to assign to the constructed point the coordinate $\frac{\mathbf{1}}{\mathbf{3}}, \frac{\mathbf{1}}{\mathbf{4}}, \dots, \frac{\mathbf{1}}{k}$ etc..

The construction of the points with the coordinates m/k does not present difficulties, for m<k. And needs some reflections for the case m>k. Here the concept of translation may be considered.-

Most difficult problem presents the construction of the points which correspond to radicals; in particular square roots, as $\sqrt{\mathbf{3}}$, etc.. This provides a more deep review of properties of right angle triangle and the circumference: the height drawn from the vertex of the right angle on the circumference is the geometric media of the projections of the edges which form the right angle on the hypotenuse: $\boldsymbol{h} = \sqrt{a_c \ b_c}$.

Further constructions lead to discussion of the nature of the numbers, rational numbers, irrational numbers, the completeness and the density property of the real line. This produce the curiosity and the interest to study real numbers.

The role of real numbers (which form the so called Real Field, R) is not a historical casualty of the development of concepts, rather it is a logical necessity.

(Their properties related to the operations of addition and multiplication depend on geometric configurations which should be completed: the Desargues configuration and the Papp's Theorem) .

Thus the conclusion is: the coordinate of a point on the axes is the distance of the point from the origin, taken with the negative sign if the point is situated on the left hand side.

Now it is not difficult to assign the coordinates of points in a plane: it is just a pair of numbers, which are the coordinate of the projections on the axis.

2.2 Equations of geometric figures on a plane

The coordinates permit to express the relation of geometric figure by means of algebraic equations.

Simple example: describe all the points which are equidistant from the coordinate axes. Obviously, it is $y = x$ and $y = -x$.

Modification of the question leads to the equations $y = kx$, and, applying parallel translation, to $y = kx + b$.

Now any straight line can be defined as the collection of points whose coordinates satisfy the linear equation: $Ax + By + C = 0$.

The intersection of two lines is related to the solution of two linear equations:

$$\begin{aligned} xa_1 + yb_1 &= d_1 \\ xa_2 + yb_2 &= d_2 \end{aligned} \tag{2.1}$$

Students are able now to solve many interesting geometric problems: to find the properties of medians, mediatrices, of heights and to construct the Euler line,

But the construction of the Cartesian system in 3dim requires the Euclidean geometry in space, which is much more complicated.

This rather long description can not be avoided in the process of construction of the Cartesian system of coordinate in a plane, since it is based completely on the theoretical fundamentals of Euclidean geometry

2.2.1 Looking for an alternative approach

Thus, mathematicians looked for another axiomatic system equivalent but more suitable system, which would produce the same result, being free of the presence of empirical experiences (unit of measurement, construction of right angle) and the objects which can not be given reasonable definitions (point, straight line, plane etc.).

We should show the students how these two methods converge to reconstruct the geometry

3 THE CONCEPT OF VECTOR IN A CARTESIAN PLANE

From the perspective of OSA, the semiotic functions not only allow us to describe the importance of geometric relations between two domains, say between Euclidean geometry and analytic geometry, but also allow us to describe other relationships between geometries and other mathematical structures (like the linear algebra) to attribute a more appropriate meaning to mathematical objects, say the vector in linear algebra. For example, we have the case of the geometric representation of vectors, which not only allows us to establish an appropriate meaning but also allows the realization of other processes such as visualization or materialization (a process that allows us to represent in an ostensive manner a mathematical object) [4-5].

3.1 Linear equation: the way to establish correspondence between coordinates of points and components of vectors

We start with an analysis of a usual representation (semiotic register) of a system of two linear equations in two unknowns in the linear algebra context: given by

$$xa_1 + yb_1 = d_1$$
$$xa_2 + yb_2 = d_2 \quad (3.1)$$

There exist various ways to represent the same system and give the corresponding interpretation.

Let say that the pairs of the coefficients $(a_1, a_2)$and ($(b_1, b_2)$are considered as the Cartesian coordinates of points A and B.

Then, considering arrows OA and OB, we can construct the diagonal OC of the parallelogram formed by the edges OA and OB. The concept of congruence of the triangles in Euclidean plane geometry suggests, that the coordinates of the point C are the sums of the coordinates of the points A and B.

There is no way to sum the points, but we can introduce the sum of arrows: $\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OC}$.

Further justification of the summation of arrows is based on the resulting forces in physics; another example is the sum of velocities.

3.2 Geometric interpretation of linear combinations of vectors in 2d

Let us start with a Cartesian plane where the coefficients of the system denoted by the same generic symbol represent the coordinates of corresponding vectors:

(3.2) $x\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + y\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$, or in the vector form $\vec{d} = x\vec{a} + y\vec{b}$.

Useful activities may be suggested in order to visualize these linear combinations for different values of coefficients: for example, the terminal points of the resulting vectors for the values of x, y from the segment [0,1] run through all points of the parallelogram with the sides formed by the vectors $\vec{a}, \vec{b}$, due to the parallelogram law for the sum of two non collinear vectors.

In turn, when the vector $\vec{d}$ is given then the sides of the parallelogram can be found by the Cramer' formulas. Here students easily solve the problem "what is the geometric meaning of vanishing determinant?": the vectors $\vec{a}, \vec{b}$ are collinear, in other words they are linearly dependent. Thus the case $\Delta \neq 0$ corresponds to the linear independence of the vectors $\vec{a}$, $\vec{b}$.

Therefore any vector $\vec{d}$, considered as a diagonal of the parallelogram, whose sides are parallel to two linearly independent directions, determines uniquely the sides of that parallelogram.

3.3 Vectors as graphical representations of Complex numbers on a Cartesian plane

One example, where the concept of Cartesian plane is required to introduce a new mathematical object, is the set of complex numbers. A complex number *z* is an expression of the form *x+iy*, where *x* and *y* are real numbers, and $i^2=-1$. We will not worry about the meaning of *i*, we are interested only in the fact that its square is *-1*. The real part of *z,* denoted by *Re z*, is the number *x* and the imaginary part of *z*, denoted by *Im z*, is the number *y*.

The set ***C*** of all complex numbers *x+iy* can be identified with the set of points *(x,y)* in the Cartesian plane, and due to this representation it is called the complex plane. The *X* axis is called the real axis and the *Y* axis is called the imaginary axis. In order to work with the complex numbers, three definitions are needed: the complex conjugate of *z*, denoted by z^{-} is the complex number *x-iy*; the module or norm of *z*, denoted by |*z*|, is the real number

$(x^2+y^2)^{1/2}$ which is the distance from the origin to the point *(x,y)* that represents *z*.

Finally, the argument of $z \neq 0$ is the angle between the positive real axis and the line through *0* and *z*, in the counter clockwise direction. The argument of *z* is denoted by *arg z* and generally it is assigned a value between *0* and $2\prod$.

There is another form to write a complex number *z*, which is known as the polar form of the complex number *z*. Let *r* be the norm of *z*, *r*=|*z*|, and θ = *arg z*, then

$z = r(\cos\{\theta\} + i \sin\{\theta\})$.This is related to another method to assign the coordinates to the points on the plane, so called the polar system of coordinates.

The set of complex numbers is similar to the set of real numbers, in the sense that there are two operations that can be applied to its elements, the sum and the product of complex

numbers. Let *z=x+iy* and *w=u+iv* be two complex numbers, then the sum of these complex numbers is given by

$$z+w= (x+iy)+(u+iv)=(x+u)+i(y+v),$$

and the product by

$$z \bullet w = (x+iy)(u+iv) = xu+ xiv +iyu+i^2 yv = (xu-yv) + i(xv+yu).$$

The set of real numbers can be seen as subset of the complex numbers, if we identified each real number

x with the complex number *(x,0)*.

It should emphasized that to the number *i* there correspond the number (0,1) and that

$$(x,y)= (x,0)+(0,y)= (x,0)+(0,1) \bullet (y,0)= x+iy.$$

The operations of sum and product of complex numbers satisfy the same properties that these operations satisfy in the set of real numbers, as being commutative, associative and the existence of the neutral elements for both operations, that are *0= (0,0)* and *1= (1,0)*, respectively.

The sum of complex numbers is exactly the same operation that the sum of vectors in the Cartesian plane.

3.4 The law of computation of the coordinates of vectors with the change of the base

Here we consider another representation of the same system (3.1) and give a new interpretation.

We should stress that what we have called coordinates of a vector, in fact, means that those are the coefficients of the decomposition of the vector with respect to, say, a standard orthogonal basis $\{\vec{e_1}, \vec{e_2}\}$ associated with the Cartesian coordinate system: $\vec{x} = x_1\vec{e_1} + x_2\vec{e_2}$.

Now, let us interpret the vector form $\vec{d} = x\vec{a} + y\vec{b}$ as a decomposition of the vector $\vec{d}$ with respect to the new basis, the meaning of x and y now is different since they serve as the coordinates with respect to this new base $\{\vec{a}, \vec{b}\}$.

Then, what is the meaning of the (semiotic register) representation of (3.1) or (3.2)?

Students have to figure out that the columns of the coefficient for each vector just mean the decomposition, for example, $\begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$ are the coordinates of the same vector $\vec{d} = d_1\vec{e}_1 + d_2\vec{e}_2$ with respect to the orthogonal base related to the Cartesian system.

What is the law to recognize when the pair of coordinates represents the same geometric vector? (because the vector is invariant: its length and direction do not depend of the choice of the base).

First, we need to express what does it mean the change of basis: the new basis vectors $\{\vec{a}, \vec{b}\}$ should be given with respect to the initial one:

(3.3a) $\vec{a} = a_1 \vec{e}_1 + a_2 \vec{e}_2$

(3.3b) $\vec{b} = b_1 \vec{e}_1 + b_2 \vec{e}_2$

If the matrix of the base change is denoted by $C = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$,

Then the initial coordinates are obtained from the new ones with the transposed matrix:

(3.4) $\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, which is just the matrix form of our initial system (3.1).

And as a consequence, the column of the new coordinates (x, y) is the result of multiplying the column with the initial coordinates by the inverse of the transposed matrix of the base change.

$\begin{pmatrix} x \\ y \end{pmatrix} = (C^T)^{(-1)} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$.

Some simple computational exercises may be performed in order to establish the formula for inverse matrix using the Cramer' rule obtained.

4 3D CONSIDERATIONS: CONCEPT OF VECTOR SPACE. AXIOMATIC APPROACH.

In teaching Linear Algebra courses, the coordinates appear as a result of the axiomatic method. How should we relate the Cartesian coordinates with these ones which are simply the coefficients of the decomposition of a vector with respect to the given bases.

As it was mentioned in Subsection 3.3, the operations of sum and product of complex numbers satisfy the same properties that these operations satisfy in the set of real numbers, as being commutative, associative and the existence of the neutral elements for both operations, that are *0= (0,0)* and *1= (1,0)*, respectively.

The sum of complex numbers is exactly the same operation that the sum of vectors.

Thus Linear Algebra states these properties as the axioms for the elements of vector space, but this development needs another presentation.

5 FINAL REMARKS

In this article we have proposed a way in which students could overcome the problem of the memorizing practice of mathematics in the classroom. This Is a set of relations (of a semiotic functions pattern) between geometric domains such as Euclidean geometry and analytic geometry as well as between the geometries and other structures such as linear algebra. Likewise, considering our teaching experience and some tests applied to students we have pointed out the importance of some empirical notions and the establishment of two relations that must be considered in the process of comprehension of geometry and linear algebra, which are not taken into account by some mathematical textbooks and teachers under the misconception to fall into informal mathematics instruction.

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