A STUDY ABOUT THE COMPREHENSION OF GEOMETRIES: THE CASE OF THE BARYCENTER

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Abstract

In this paper we share our educational experience in teaching methodologies implemented in the traditional mathematical courses for carriers of chemical engineers as well as the results of evaluations of students learning based on the methodology elaborated in the frames of the Onto Semiotic Approach Theory (OSA). The latter consists of a qualitative study carried out at the end of a mathematics course. The results of evaluations reveal that most students who have no previous knowledge of Euclidean geometry, solve the given task operatively (and some are unable to resolve it) and no proper meaning is attributed to certain elements of the analytic geometry. However, students who have prior knowledge of Euclidean geometry show a better performance.

Keywords: educational experience, teaching methodologies, Ontosemiotic approach, Analytic geometry.

1 INTRODUCTION

In this paper, we present results from a qualitative study carried out at the end of a mathematics course in which have been analyzed the comprehension of Analytic Geometry (AG) by a group of university students. Based on the assumption that students could achieve meaningful learning if some topics from the Euclidean Geometry (EG) are taken into account before the AG course, we propose the problem "Given the coordinates of a system of three particles $P_1(x_1, y_1)$,$P_2(x_2, y_2)$ y $P_3(x_3, y_3)$ on a Cartesian plane, find the barycenter of the system", to study how students consider some elements from the Euclidean geometry to obtain the barycenter "G " in the context of AG without the operational use of tools like the expression $G\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$.

Theoretical elements from the Ontosemiotic Approach, OSA (Godino, Batanero & Font, 2007; Font, Planas y Godino, 2010), are employed to describe the mathematical activity carried out by the students. In OSA, mathematical activity plays a central role and is modeled in terms of systems of operative and discursive practices. From these practices, the different types of related mathematical objects (language, arguments, concepts, propositions, procedures, and problems) emerge, building cognitive (personal perspective) or epistemic (institutional perspective) configurations among them. In this context, problems promote and contextualize the activity; languages (symbols, notations, graphics, etc.) represent the other entities and serve as tools for action; arguments justify the procedures and propositions that relate the concepts.

For the qualitative study we rely upon the analysis technique that comes from the ontosemiotic approach (Malaspina, 2007; Malaspina & Font, 2010) that allow us to describe in a systematic way the cognitive configurations of the students (problem-situation, representation, concepts, properties, procedures and arguments) obtained from the solution to the task, and make a comparison with a reference epistemic configuration. Two groups of ten students each, a control group (no prior knowledge of EG) and a pilot group (with previous knowledge of EG) were analyzed. The results reveal that most students who have no previous knowledge of EG, solve the given task operatively (and some are unable to resolve it) and no proper meaning is attributed to certain elements of the AG. However, students who have prior knowledge of EG show a better performance.

2 OBJECTIVE OF THE RESEARCH

In the explanation of a teacher in the classroom and mathematics texts, some visual representations, virtual objects, etc. commonly are used. Now, frequently the picture suggests to the reader to complete the information. Conversely, the diagrams used by students to solve the task, are the

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success or outcome of a complex mathematical activity in which concepts are used as well as properties, methods and arguments, and they are involved in the game in various processes, including those of significance and visualization. If the iceberg metaphor is used, the ostensive student production is only part of the complex web of practices, use of laws, procedures, processes, etc.. that have been enabled to give the answer to the required task.

The research presented is related to the comprehension of the AG in which diagrams are the most visible part. Specifically, studies of the complexity of the mathematical activity (characterized in terms of practices, content and mathematical processes) while solving the task, related to the solution of the problem of a system of three particles, and analysis of the diagrams presented, in which the barycenter involved, play an essential role.

The barycenter supports a conceptualization, where it is considered as the point of intersection of the medians of a triangle and as the equilibrium point of the action of forces, in the latter case it plays an intuitive role. The most common problems are those in which the barycenter is understood as the intersection point of the medians. In this research, to better highlight the role of diagrams, we selected a problem in the case of a system or three particles where the most appropriate conceptualization is in terms of the point of intersection of the medians of a triangle

3 THEORETICAL BACKGROUND

This paper describes a study about the comprehension of analytic geometry through the solving a problem related to the calculation of the coordinates of the barycenter of a system of three particles from the theoretical framework of Ontosemiotic Approach, OSA, [1-2]. The present research is not focused on analytic geometry in general but, rather, on a specific topic, the barycenter coordinates.

In OSA, the mathematical activity plays a central role and is modeled in terms of systems of operative and discursive practices. From these practices, the different types of related mathematical objects (language, arguments, concepts, propositions, procedures, and problems) emerge, building cognitive (personal perspective) or epistemic configurations (institutional perspective) among them. In this framework, problems promote and contextualize the activity; languages (symbols, notations, graphics, etc.) represent the other entities and serve as tools for action; arguments justify the procedures and propositions relate the concepts.

The objects that appear in mathematical practices and those which emerge from these practices depend on the “language game” (Wittgenstein, 1953) in which they participate and might be considered from the five facets of dual dimensions: personal/institutional, unitary/systemic, expression/content, ostensive/non-ostensive, and extensive/intensive. Both the dualities and objects can be analyzed from a process-product perspective, so that if we have an argument an argumentation process have occurred, if we have an ostensive object a process of representation have occurred, and so on. Instead of giving a general definition of process, the OSA opts to select a list of processes that are considered important in mathematical activity.

4 METHODOLOGY

For the qualitative study we rely on technical analysis of OSA [3-4], for describing systematic cognitive configurations (problem-situation, representations, concepts, properties, methods and arguments) of students, to solve the task given and perform a comparison with an epistemic reference configuration Table 1.

In order to evaluate students learning we rely on the assumption that students could achieve meaningful learning if some topics from the EG are taken into consideration previously, proposed the problem that is very useful in applications: “Given the coordinates of a system of three particles on the plane, find the coordinates of the Center of Masses (barycenter) of the system”. The objective was to study how students take into consideration some elements from the EG to obtain the barycenter “G” in the context of AG as well as to explain the operational use of usual formulas.

The participants were 25 students between 19 y 20 years old that were coursing the third semester in the science and engineering faculty of the Morelos state university, Mexico. These students had been approved the basic course in mathematics where some topics on analytic geometry were analyzed

The task proposed to the students was the problem of calculating the coordinates of the barycenter of a system of three particles located on a plane. This task was selected because the activity carried out by students consider the essential elements of understanding analytic geometry, that is, in the solving problem appear objects like frame of reference, coordinates, line, etc. The problem was: Find the coordinates of the center of gravity of three particles located at the three points with arbitrary coordinates.

4.1 Practice, Epistemic configuration and processes

To solve the task of the system of three particles, the expert solver must carry out a mathematical practice that involves reading the text of the task and the production of a text with the steps performed to solve the task, and that will be the result of some series of reasoning subject to mathematical rules. Expert solves the task in accordance with a procedure that has been studied in a typical classroom session. First, considering the coordinates of the particles, the coordinates of mid point of every pair of points are calculated. Subsequently, expert determines the equation of two of the three straight lines that pass trough each mid point (there are three points, one in each side of the triangle) and the corresponding opposite vertex of the triangle. Thus, a system of two equations with two unknowns is obtained, whose solution corresponds to the coordinates of the barycenter.

In Table 1 epistemic setting enabled in solving the task is presented, it was obtained after an expert triangulation applied to a first proposed by the authors:

Table 1. Epistemic configuration of reference. Expert solution of the task.

Mathematical objects	Characteristics
Language	—Diagrammatic representation, where there are involved: • Mathematical terms: points, straight line, mid point, medians, triangle, intersection, system of equations, solution • Symbols: Points "A, B, C and G ", coordinates "$(x_1, y_1), (x_2, y_2)$ and ((x_3, y_3))", ,
Concepts	—Cartesian plane, coordinates, segment, medians, system of equations barycenter.
Properties	—the intersection of two medians of the triangle correspond to the barycenter of the triangle —an equation of a straight line can be obtained from the coordinates of two points: the mid point of the segment, that goes from one vertex of the triangle to another, and a vertex of the triangle opposite to the mid point —the solution to the system of equations is unique
Procedure	—determine the algebraic equations of the straight lines that contain each median —solve the system of equations
Argument	—Thesis: the coordinates of the barycenter are calculated by means of a solution of a system of equations. The generalized solution of the three linear equations that correspond to each median allows to calculate the barycenter by means of the expression $G\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$ Reasons • The intersection of the medians correspond to the solution of the system of the equations • The barycenter is a unique point, this can be demonstrated if we consider the third median.

The elements of this configuration (from the process-product perspective) involve processes such as visualization and idealization, among others. According to [5] we can speak of primary visual configurations of objects, when these processes involve visualization. The task of the system three particles can be considered a visual task in correspondence with the two proposed features in [5]: a) Communication of form, its components and structure of space objects, or objects imagined (thought or ideal) b) Communication from the relative position of objects in space. Indeed, both are true:

a) The task communicates visual information. Components (particles) and shape (triangle) are explained and the student is expected to carry out a process of idealization and understand that the barycenter can be considered as a point (ideal object) which is the point of intersection of the medians. This textual information suggested by the task must be transformed by the student in a representation on the plane (two-dimensional).

b) The task communicates the relative position of objects in space. For solving the problem, the student must specify a frame on which the information of the relative position must be located.

Similarly, most of the concepts, procedures, representations, properties and arguments of this epistemic configuration can also be considered as "visual", according to the characterization proposed [5].

5 RESULTS

In Fig. 1(a-c) we present the schema elaborated and the solution of the student. The task is solved correctly but it is carried out in an operative form. The solution considers the three particular coordinates A, B and C. In the solution of the task the student considers the formula $(x, y) = \left(\frac{x_A + x_B + x_C}{3}, \frac{y_A + y_B + y_C}{3}\right)$ directly without demonstrating its validity.

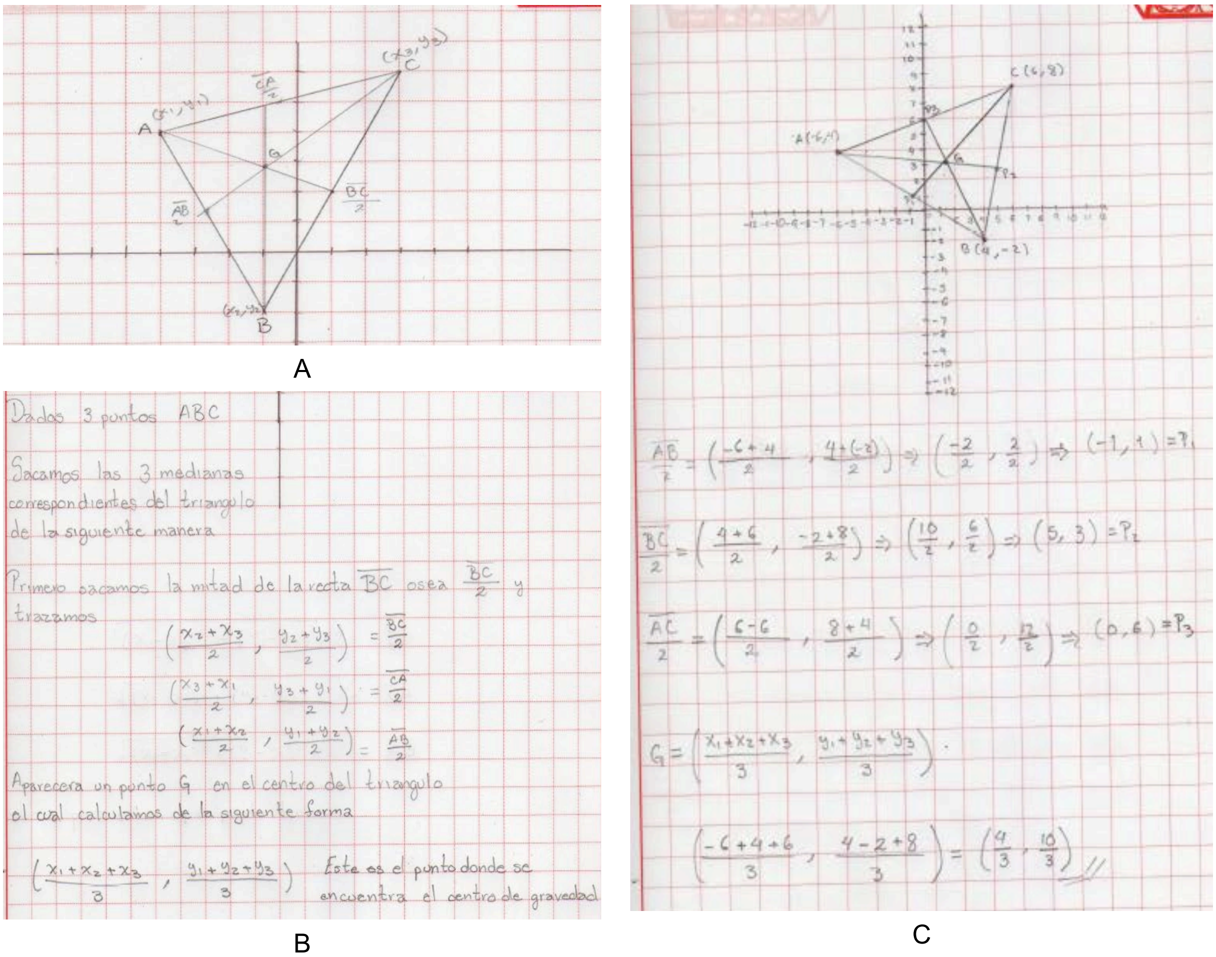

Figure 1. The solution of the student to the task.

We can see that the student starts from de calculation of the midpoints of the edges of the triangle and skips a certain procedure, finally, only the resultant formula is represented. Then, the student just replaces the values of the coordinates in the formula and obtains the coordinates of the barycenter. In Table 2 we present the cognitive configuration of this student.

Table 2. Cognitive configuration of reference. Expert solution to the task.

Mathematical objects	Characteristics
Language	—Diagrammatic representation: —Mathematical terms: points, straight line, midpoint, medians, triangle —Symbols: Points "A, B, C and G ", coordinates "$(x_1, y_1), (x_2, y_2)$ and ((x_3, y_3))",
Concepts	—Cartesian plane, coordinates, segment, barycenter.
Properties	—the x coordinate of a particle can be added to another x coordinate, the same for the y coordinate.
Procedure	—identify the x and y coordinates of each particle, Fig. 1(c). —replace the values on the formula. —performing operations.
Argument	—Thesis: the coordinates of the barycenter is calculated by means of the formula $\left(\frac{x_A + x_B + x_C}{3}, \frac{y_A + y_B + y_C}{3}\right)$ Reasons: • The barycenter is a point located at the center of the triangle, Fig. 1(b).

Table 2 shows that student solve the problem operatively because only replaces the values of the coordinates in the formula.

With respect to the process carried out by the student we argue that a process of representation have occurred because the three points, a frame of reference and the barycenter were represented. We consider that both visualization and argumentation did not occur because the student did not consider relation between EG and AG related to any triangle, only the barycenter property has been taken into account.

6 TEACHING METHODOLOGIES

As a consequence of the results obtained in our research about the comprehension of analytic geometry on the basis of Euclidean geometry axioms we proposed two teaching methodologies, which were implemented in the traditional mathematical courses for carriers of chemical engineer. By means of the proposed methodologies, the problem to find the barycenter of a system of three and four particles located on the plane is solved. The first takes into account the physical notion of equilibrium of three forces, that is, the action of three forces on the same point is equal to zero. The second methodology, considers the axiom of linear independence to find the barycenter of a parallelogram with particles located in their vertices.

Under the EOS perspective, the methodology corresponds to a practice from which the barycentre as mathematical object emerges. The properties of the barycenter pointed out in the context of Euclidean geometry related to the triangle and the parallelogram are employed to validate the results obtained by means of the vectorial algebra procedures.

The methodologies are described in the following subsections.

6.1 The barycentre on the basis of the physical notion of force

In the Fig. 2 we show three forces $\vec{a}$, $\vec{b}$ and $\vec{c}$ acting on the same point "$\mathbf{0}$". The action of $\vec{a}$ and $\vec{b}$ gives the resultant force $\vec{\tau}$. This latter force cancels the force $\vec{c}$ to obtain the equilibrium of forces.

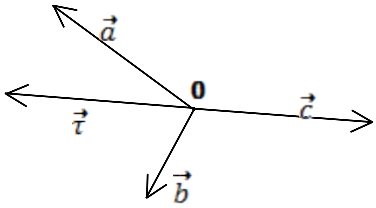

Figure 2. The sum of three vectors on a same point $\mathbf{0}$.

By the other side, we suggest to students the configuration of a system of three particles A , B and C as seen in the Fig. 3. The position of the particles on the plane with respect to a frame of reference "$\mathbf{0}$" are described by vectors $\vec{r}_A$, $\vec{r}_B$ and $\vec{r}_C$. At the same time, we point out to students the description of the position of the particles with respect to a point G (the barycentre) located by the vector $\overrightarrow{\mathbf{0}G}$ through vectors $\overrightarrow{GA}$, $\overrightarrow{GB}$ and $\overrightarrow{GC}$.

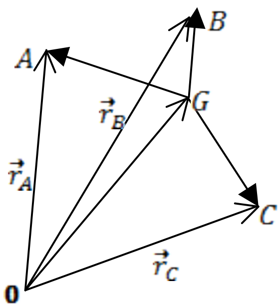

Figure 3. The system of three particles and the barycentre G .

The positions of these particles are described through the following vector sum:

$$\overrightarrow{GA} = \vec{r}_A - \overrightarrow{\mathbf{0}G}$$

$$\overrightarrow{GB} = \vec{r}_B - \overrightarrow{\mathbf{0}G}$$

$$\overrightarrow{GC} = \vec{r}_C - \overrightarrow{\mathbf{0}G}$$

Adding these vectors and taking into account the idea of equilibrium of forces we obtain: $0 = \vec{r}_A + \vec{r}_B + \vec{r}_C - \mathbf{3}\overrightarrow{\mathbf{0}G}$, because $\overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} = \mathbf{0}$. That is, the position of the point G is described by means of the vectorial equation:

$$\overrightarrow{\mathbf{0}G} = \frac{\mathbf{1}}{\mathbf{3}(\vec{r}_A + \vec{r}_B + \vec{r}_C)}$$

That according to Euclidean geometry corresponds to the position of the barycentre of a triangle which vertices are located in the positions $\vec{r}_A$, $\vec{r}_B$ and $\vec{r}_C$.

6.2 The barycentre on the basis of linear independence

In the second methodology we consider the axiom of linear independence of a set of base vectors, say $\vec{b}$ and $\vec{c}$, with the possibility of increasing of the dimension of the employed vector space. In general, the coordinates of the parallelogram are obtained from the linear combination of the basis vectors.

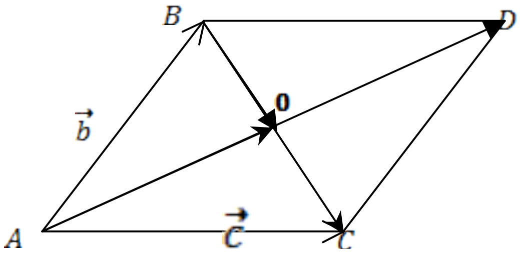

Figure 4. The barycentre of a system of four particles.

The barycentre of a system of four particles A, B, C and D located at the corners in a parallelogram, see Fig. 4, is calculated through the scalar product $\overrightarrow{B\mathbf{0}} = \lambda\overrightarrow{BC}$ and $\overrightarrow{A\mathbf{0}} = \mu\overrightarrow{AD}$ (with $\lambda, \mu < \mathbf{1}$) and the vector sum as follows:

$$\vec{b} + \overrightarrow{B\mathbf{0}} = \overrightarrow{A\mathbf{0}}$$

$$\vec{b} + \lambda\overrightarrow{BC} = \mu\overrightarrow{AD}$$

$$\vec{b} + \lambda(\vec{c} - \vec{b}) = \mu(\vec{b} + \vec{c})$$

Because $\overrightarrow{BC} = \vec{c} - \vec{b}$ and $\overrightarrow{AD} = \vec{b} + \vec{c}$, see Figure 3. Solving the latter expression for the coefficients we obtain $\lambda = \frac{\mathbf{1}}{\mathbf{2}}$ and $\mu = \frac{\mathbf{1}}{\mathbf{2}}$. We can demonstrate that the point A correspond to the barycentre by taking other vectors, for example $\overrightarrow{DB}$ and $\overrightarrow{DC}$ in terms of the basis vectors $\vec{b}$ and $\vec{c}$.

7 CONCLUSIONS

The results of the test reveals that most students who have no previous knowledge of EG, solve the given task operatively (and some are unable to resolve it) through the use of a *mathematical formula* and no proper meaning is attributed to certain elements of the AG. However, students who have prior knowledge of EG show a better performance.

The comparison between de epistemic and cognitive configuration shows that it is necessary take into account some elements from the EG to solve problems in the context of AG. The mathematical object, barycentre, is common both in EG and AG so that this enables the use in the context of AG of some properties that coming from the EG. This constitutes the central part of our proposed teaching methodologies. The visualization and the establishment of relation between the properties described in the context of EG and some steps in the solution of the task in the AG context enables students generate a meaning of the mathematical objects.

These pedagogical innovations concern mostly the so called problem-based learning which has been enhanced with the original didactic materials providing visualization and geometric interpretations of abstract mathematical objects. Similar results have been obtained on the evaluation of some topics of the courses on Probability and Statistic depending on the implementation of didactic material providing some technology of visualization.

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