VISUALIZATION OF NON-OSTENSIVE MATHEMATICAL OBJECTS AS GEOMETRIC INTERPRETATIONS OF SEMIOTIC REGISTERS IN ALGEBRAIC SETTING

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Abstract

In this presentation we describe our experience and learning methodology in the teaching the standard Linear Algebra course for the students who had no opportunity to obtain the proper knowledge at their school level mathematical education. The initial stage of introducing the fundamental notions through exploratory activities has been proposed and the didactic materials have been elaborated, according to the theory developed as Onto-Semiotic Approach (OSA), demonstrating, in particular, how mathematical objects emerge from mathematical practices, and what interpretations could be suggested with respect to different semiotic representations.

Keywords: Determinants, Ontosemiotic approach, visualization.

1 INTRODUCTION. THEORETICAL FRAMEWORK

The mathematical ontology that is proposed in the Ontosemiotic Approach, along with the notion of semiotic function, can contribute to achieve the main goal of mathematics instruction. Ecology of meanings provides an appropriate way to realize different types of analysis on the different didactic levels in the production and communication of mathematics. [1].

The study of determinants in the theoretical aspects presents serious problems for the students of the Linear Algebra course. Being aware of the crucial role which the visualization plays in learning demand to achieve the comprehension of a concept, we suggest some geometric interpretation which may be used to produce images that accompany the mathematical ideas employed in different context.

The concept of the determinants naturally emerges in the process of solving linear equations. Being involved in different contexts and situation problems, the traditional algebraic representation of a system of linear equations takes different semiotic representations such as the so called matrix form or the form of linear combination of vectors. Interpretation of every one of these semiotic registers usually depends on the contexts, for example, a vector form register can be treated as the linear dependency of the elements of a vector space or as the representation of the image of an element under a linear transformation (Grossman, 1987, p.273), as well, as a representation of the coordinates with respect to different bases. The theory of OSA allows to study difficulties related to the semiotic function (Font, et al. 2007), which students encounter in their independent studies

In our exploratory stage of the course we rely upon the OSA theory: "the new primary objects appear as a result of mathematical practices and become institutional primary objects due, among other things, to processes of institutionalization" [1].

OSA considers the notion of emergence useful to describe not only mathematical activity at the individual level (learning processes) but also the progressive social construction of mathematical knowledge from a historical and epistemological point of view.

Further we will provide the most relevant theoretical statements which we have taken into account in the process of elaborating of our didactic materials, and which can be found in [1],[2] [3].

Primary objects can be regarded as 'being' in mathematical practices is therefore related to the ostensive/non-ostensive duality

However, there are several reasons why, in mathematical discourse, a distinction is made, whether implicitly or explicitly, between ostensive representations and non-ostensive objects.

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In mathematics discourse it is possible to talk about ostensive objects representing non-ostensive objects that do not exist. As well, there are different representations which are regarded as representations of the same mathematical object.

The ontological approach considers that in the context of classroom mathematics, a language game is developed which leads students to regard ostensive material representations as being different from both the thoughts which people have when using them and the objects they represent.

In mathematical discourse a language game is produced that leads to the emergence of an object, one that is not considered to be ostensive, cannot be identified with any of the primary objects of the configuration, and which, moreover, is considered to be the referent for that configuration when considered as a whole

This association would explain how it is possible to consider that the object can be defined in different, equivalent ways, or that it can be represented by different representations, etc.

When performing mathematical practices we make reference to an already-known socio-epistemic configuration of primary objects and, as a result, access a new configuration of primary objects in which one (or more) of these objects was previously unknown. This leads to the emergence of the object that serves as the global referent. This emergence is due to the combined effect of different dualities.

The personal/institutional duality provides a way of thinking about how primary mathematical objects can be regarded as having 'being' in mathematical practices. Such objects may participate as either personal or institutional objects and, depending on the language game, they may shift from being personal to institutional.

2 EXPLORATION THROUGH COMPUTATIONAL PRACTICE

The aim of our elementary introduction to the linear algebra course is to accompany students to discover the most important concepts: determinant, inverse matrix, change of base, linear composition, independence, vector product, areas and volumes, linear transformations, different kinds of matrices, the trace of a matrix, etc..

Let us emphasize that at the initial exploratory practice we shall relay upon on the previous knowledge of students only, keeping in mind the lack of systematic geometric study.

2.1 Determinant as a mathematical object emergent in the process of computational (operative) practice

We start with an analysis of a usual representation (semiotic register) of a system of two linear equations in two unknowns in the linear algebra context:

(2.1) $$\begin{aligned} xa_1 + yb_1 &= d_1 \\ xa_2 + yb_2 &= d_2 \end{aligned}$$

The necessity to introduce the concept of determinant of the given system appears in the process of solving the system by the method of elimination of variables since the coefficient of each variable in the reduced form of the system: x($a_1\, b_2 - a_2\, b_1$)=($d_1\, b_2 - d_2\, b_1$),

y($a_1\, b_2 - a_2\, b_1$)=($d_2\, a_1 - d_1\, a_2$), determines whether the system possesses a unique solution, i.e. it provides a criterion. The determinant is denoted by $\Delta = (a_1\, b_2 - a_2\, b_1)$ and, furthermore, is represented with the help of the table of coefficients of the system (2.1):

(2.2) $$\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1\, b_2 - a_2\, b_1 .$$

As a result, one can state: in the case $\Delta \neq 0$ there exists a unique solution represented in the form $x = \frac{\Delta_x}{\Delta}$, $y = \frac{\Delta_y}{\Delta}$, well known as the Cramer' rule, $\Delta_x = \begin{vmatrix} d_1 & b_1 \\ d_2 & b_2 \end{vmatrix}$, $\Delta_y = \begin{vmatrix} a_1 & d_1 \\ b_1 & d_2 \end{vmatrix}$.

In what follows we propose different environment (situation-problems) in order to give accordingly the interpretations of this system so that the students could discover a correspondent geometric meaning for this new object.

2.2 Geometric interpretation of linear combinations of vectors in 2d

There exists various ways to represent the same system and give the corresponding interpretation.

Let us start with a Cartesian plane where the coefficients of the system denoted by the same generic symbol represent the coordinates of corresponding vectors:

(2.3) $x\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + y\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$, or in the vector form $\vec{d} = x\vec{a} + y\vec{b}$.

Useful activities may be suggested in order to visualize these linear combinations for different values of coefficients: for example, the terminal points of the resulting vectors for the values of x, y from the segment [0,1] run through all points of the parallelogram with the sides formed by the vectors $\vec{a}, \vec{b}$, due to the parallelogram law for the sum of two non collinear vectors.

In turn, when the vector $\vec{d}$ is given then the sides of the parallelogram can be found by the Cramer' formulas. Here students easily solve the problem "what is the geometric meaning

of vanishing determinant (2.2)?": the vectors $\vec{a}, \vec{b}$ are collinear, in other words they are linearly dependent. Thus the case $\Delta \neq 0$ corresponds to the linear independence of the vectors $\vec{a}, \vec{b}$.

Therefore given any vector $\vec{d}$, considered as a diagonal of the parallelogram, whose sides are parallel to two linearly independent directions, determines uniquely the sides of that parallelogram.

2.3 Area of a parallelogram in 2dim setting

The natural question arises, how to calculate the area of such a parallelogram in general case.

Usually the only formula students know is the "product of the base by height".

Introducing into the consideration the angle between two vectors, they can obtain a more symmetric formula for the area of the parallelogram with the sides $x|\vec{a}|$and $y|\vec{b}|$, namely the area of this parallelogram A =Area(x,y, $\vec{a}, \vec{b}$)= $x|\vec{a}|\,y|\vec{b}|\sin\alpha$.

There are different approaches to calculate the value of A, nevertheless we suggest to use the trigonometric formulas and the basic expression for the scalar product $(\vec{a}, \vec{b}) = |\vec{a}||\vec{b}|\cos\alpha$.

To simplify computation we denote $\vec{u} = x\vec{a}$ and $\vec{v} = y\vec{b}$,

Thus we have that the square of the area is $A^2 = (\vec{u}, \vec{u})(\vec{v}, \vec{v})(\sin\alpha)^2 =$

(2.4) $(\vec{u}, \vec{u})(\vec{v}, \vec{v})\ (1 - [(\vec{u}, \vec{v})^2 / (\vec{u}, \vec{u})(\vec{v}, \vec{v})]$.

Or $(u_1^2 + u_2^2)(v_1^2 + v_2^2) - (u_1v_1 + u_2v_2)^2 = (u_1v_2 - u_2v_1)^2$(as the result of direct calculation and simplification of the left hand side expression).

The final expression is just the square of the determinant whose entries are the coordinates of the vectors –sides of the parallelogram.

Finally we obtain: $A = xyArea(1,1,\vec{a},\vec{b}) = xy\,|\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}|$, and it is easy to see that the absolute value of the determinant (2.2) represents the area of the parallelogram corresponding to values x=1, y=1.

Now students may explore different cases, say x=3, y=1; x=1,y=2, x=3, y=2, etc, to visualize

the additive properties of the area, especially useful for the students with the lack of elementary knowledge of similarity of triangles (i.e. they should produce images that accompany the mathematical ideas employed).

The same problems for 3-dimensional case will be considered in Section 3, where, as a result of computational practice, the new mathematical object will emerge, known as vector product.

2.4 The law of computation of the coordinates of vectors with the change of the base

Here we consider another representation of the same system (2.1) and give a new interpretation.

We should stress that what we have called coordinates of a vector, in fact, means that those are the coefficients of the decomposition of the vector with respect to, say, a standard orthogonal basis $\{\vec{e}_1, \vec{e}_2\}$ associated with the Cartesian coordinate system: $\vec{x} = x^1\,\vec{e}_1 + x^2\,\vec{e}_2$.

Now, let us interpret the vector form $\vec{d} = x\,\vec{a} + y\,\vec{b}$ as a decomposition of the vector $\vec{d}$ with respect to the new basis, the meaning of x and y now is different since they serve as the coordinates with respect to this new base $\{\vec{a}, \vec{b}\}$.

Then, what is the meaning of the (semiotic register) representation of (2.1) or (2.3)?

Students have to figure out that the columns of the coefficient for each vector just mean the decomposition, for example, $\begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$ are the coordinates of the same vector $\vec{d} = d_1\,\vec{e}_1 + d_2\,\vec{e}_2$ with respect to the orthogonal base related to the Cartesian system.

What is the law to recognize when the pair of coordinates represents the same geometric vector? (because the vector is invariant: its length and direction do not depend of the choice of the base).

First, we need to express what does it mean the change of basis: the new basis vectors $\{\vec{a}, \vec{b}\}$ should be given with respect to the initial one:

(2.5a) $\vec{a} = a_1\,\vec{e}_1 + a_2\,\vec{e}_2$

(2.5b) $\vec{b} = b_1\,\vec{e}_1 + b_2\,\vec{e}_2$

If the matrix of the base change is denoted by $C = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$,

Then the initial coordinates are obtained from the new ones with the transposed matrix:

(2.6) $\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, which is just the matrix form of our initial system (2.1).

And as a consequence, the column of the new coordinates (x, y) is the result of multiplying the column with the initial coordinates by the inverse of the transposed matrix of the base change.

$\begin{pmatrix} x \\ y \end{pmatrix}$=(C ^{T})$^{(-1)}\begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$.

Some simple computational exercises may be performed in order to establish the formula for inverse matrix using the Cramer' rule obtained in Subsection 2.1.

2.5 Interpretation of the matrix form as a linear transformation

Let us draw a picture which represents the vectors (2.5a), (2.5b). and (2.6).

Now let us introduce the new vector $\vec{u}$ with the same coordinates as the vector $\vec{d}$, in its decomposition as the diagonal of the parallelogram, i. e., $\vec{u} = x\vec{e}_1 + y\vec{e}_2$.

Then there can be given a very important construction of the transformation ϕ, which transforms the vectors { $\vec{e}_1, \vec{e}_2, \vec{u}$ } into its images which will occupy the position determined by the vectors $\{\vec{a}, \vec{b}, \vec{d}\}$. Such transformations are called linear, and it is enough to determine only the images of bases vectors, in this case, according to the formulas (2.5a), (2.5b).

Then due to linearity, which means, $\phi($ x $\vec{e}_1 + y\vec{e}_2$)= xϕ ($\vec{e}_1$)+y ϕ($\vec{e}_2$)= $x\vec{a} + y\vec{b}$ we have

$\phi(\vec{u}) = \vec{d}$, i.e. the diagonal of the parallelogram with the sides on the bases vectors is transformed into the diagonal of the parallelogram with the sides on the vectors-images of the bases vectors. Thus the matrix form (2.6) expresses the coordinates of the transformed vector with respect to the initial bases, and the matrix is called the matrix of the transformation ϕ with respect to this base.

Various exercises may be suggested in order to clarify the meaning of the matrix corresponding to particular types of linear transformations, which reflect the geometric properties of Euclidean geometry (orthogonal, symmetric, skew-symmetric, diagonal)

As well, the more general meaning of the determinant of the matrix of the transformation may be described easily, which leads to series of applications in mechanics, where the interpretation of the trace of matrix can be given in terms of infinitesimal linear transformations.

3 3D CONSIDERATIONS

In Section 2 we have achieved a significant comprehension of the fundamental concepts in linear algebra on the level which is called in EOS extensive, o personal.

We have to generalize these notions in order to arrive to the institutional level (extensive intensive duality).

3.1 Solution of system of two homogeneous equations with three variables

Let us find the solutions of a homogeneous system of two equations with three variables as a useful exercise for the further subsections:

$$\begin{matrix} xa_1 + yb_1 + zc_1 = 0 \\ xa_2 + yb_2 + zc_2 = 0 \end{matrix},$$

We can apply the Cramer' rule, assuming that $\Delta = (a_1 b_2 - a_2 b_1)$ is not vanishing.

One can obtain $x = \dfrac{\begin{vmatrix} -c_1 & b_1 \\ -c_2 & b_2 \end{vmatrix}}{\Delta}$ z, $y = \dfrac{\begin{vmatrix} a_1 & -c_1 \\ a_2 & -c_2 \end{vmatrix}}{\Delta} z$, where z can be given any value.

In particular, if we assign $z = \Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ then we obtain a particular solution.

(3.1) $\{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}, -\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}, \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}\}.$

The emerging mathematical object constructed by means of these three 2x2 determinants formed by the coefficients of the system further will be recognized as a result of the composition of two vectors and it will be given the name “vector product”.

As well, here we may arrive to the concept of linear space formed by the set of all solutions, taking $z = \Delta t$, where the parameter t can take any real value.

3.2 Area of the parallelogram constructed on the two linearly independent vectors in 3dim.

Let us return to the area of the parallelogram in the Sec 2 , 2.4, for the three dimensional case. Applying the formula (2.4) to the case with the three corresponding coordinates

$$(u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) - (u_1 v_1 + u_2 v_2 + u_3 v_3)^2 =$$
$$(u_1 v_2 - u_2 v_1)^2 + (u_1 v_3 - u_3 v_1)^2 + (u_2 v_3 - u_3 v_{21})^2$$

Comparing with the determinants in the (3,1) one can continue the equality as

$$\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}^2 + \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}^2 + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}^2 = (\vec{\zeta}, \vec{\zeta})$$

Thus, the area of the parallelogram is expressed in 3d case as the length of the vector $\vec{\zeta}$ with the coordinates calculated by the rule (3.1) applied to the coordinates of vectors $\vec{u}$ and $\vec{v}$.

At the same time the corresponding system, which gave rise to this vector $\vec{\zeta}$, expresses the vanishing of the inner products of this vector with each of the two vectors $\vec{u}$ and $\vec{v}$, the geometric meaning being orthogonal to $\vec{u}$ and $\vec{v}$. (the peculiar sign of the second coordinate gives the preference in orientation of the three vector as a right-oriented triple).

The vector $\vec{\zeta}$ is given name “vector product” and is denoted by [$\vec{u}$, $\vec{v}$], or $\vec{u} \times \vec{v}$ (cross-product).

We have to stress that the construction of a scalar (inner) product is valid in the case of the Euclidian space, nevertheless the similar meaning of the determinant for an affine space takes place although there is no concept of volume.

The geometric meaning applied to the usual interpretation and visualization valid for analytic geometry is beyond of the scope of this article.

3.3 Solution of general system of three equations with three variables.

Now the students are in position to pass to the case of the system on three unknowns

$$\begin{aligned} xa + yb + zc &= d \\ xa_1 + yb_1 + zc_1 &= d_1 \\ xa_2 + yb_2 + zc_2 &= d_2 \end{aligned} \qquad (3.2)$$

Following the way of 2.1 we can write directly the solutions of the last two equations $x = \dfrac{\begin{vmatrix} d_1 - c_1 z & b_1 \\ d_2 - c_2 z & b_2 \end{vmatrix}}{\Delta}$, $y = \dfrac{\begin{vmatrix} a_1 & d_1 - c_1 z \\ a_2 & d_2 - c_2 z \end{vmatrix}}{\Delta}$, with $\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$,

and substitute these expressions for unknowns x and y into the first equation, that yields

after simplifications

$$\left(a\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} - b\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} + c\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \right) z = (T)$$ (T stands for the free term).

As in previous exploration the coefficient of z determines whether there exits a unique solution, thus this coefficient is called the "determinant"

(3.3) $$\Delta_{3\times 3} = \begin{vmatrix} a & b & c \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = a\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} - b\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} + c\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

Furthermore, after ingenious efforts of rearranging the right hand side term (T) one arrives to

$\Delta_{3\times 3} z = a\begin{vmatrix} b_1 d_1 \\ b_2 d_2 \end{vmatrix} - b\begin{vmatrix} a_1 d_1 \\ a_2 d_2 \end{vmatrix} + d\begin{vmatrix} a_1 b_1 \\ a_2 b_2 \end{vmatrix}$, which now can be written as $\Delta_{3\times 3} z = \Delta_z = \begin{vmatrix} a & b & d \\ a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \end{vmatrix}$.

Thus the new mathematical object, determinants 3x3, emerge in the process of operative practice. Much of work may be suggested to obtain the standard properties of determinants.

3.4 The volume of the parallelepiped constructed with three linearly independent vectors

Here naturally emerge the concept of triple product and vector products with their geometric interpretations.

Analyzing the formula (3.3) which express the rule of calculating of the 3x3 determinant, one can conclude that algebraically it coincides with the coordinate expression for the inner product of the vector $\vec{w}$ with the coordinates as the entries of the first raw, with the vector product of the two vectors formed with the coordinates of the second and the third raw correspondingly, say $\vec{u}$ and $\vec{v}$.

We have $\Delta = \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (\vec{w}, [\vec{u}, \vec{v}])$

The absolute value is expressed as $|\vec{w}|\,|[\vec{u}, \vec{v}]| \cos\theta$, the geometric meaning of the second factor being the area of the parallelogram, as the base, constructed on the two linearly independent vectors, and the product of others factors has the geometric meaning as the height of the parallelepiped traced toward the base.

Thus the absolute value of the 3x3 determinant expresses the volume of the corresponding parallelepiped, meanwhile its sign is related to the orientation of the triple.

As well, there naturally emerges the concept of the triple product $(\vec{w}, \vec{u}, \vec{v}) = (\vec{w}, [\vec{u}, \vec{v}]) = ([\vec{w}, \vec{u}], \vec{v})$, which can be proven as an exercise.

As a result the 3x3 determinant is related to the expression of the volume of the corresponding parallelepiped constructed on the three vectors with the coordinates formed by entries of the rows of the determinant, as well as to the orientation of this triple of vectors.

We have to emphasize that this result is achieved due to construction of a scalar (inner) product, which is valid only in the case of the Euclidian space, nevertheless the similar meaning of the determinant for an affine space takes place (see 3.6).

3.5 Semiotic conflict

Let us note that one can not give an interpretation of the results of 3.4. similar to the constructions considered in Section 2.

There is no sense to consider this 3x3 system as a linear combination of vector-columns in order have the interpretations as in Section 2.

Nevertheless the desired geometric interpretation can be applied to the vector-columns, because algebraically the triple product of these vector-columns will give the same result.

This fine result is based on the property that the determinant of the matrix is equal to the determinant of the transposed matrix.

Thus the similar criteria for the linear dependence-independence can be elaborated easily, giving the proper geometric interpretation as a form of a visualization.

It may be said here that mathematically this conflict is responsible for the more caution interpretation of the entries of the matrix of the system: meanwhile the columns form the elements of vectors space, the nature of the row entries is different: they are co-vectors and form the dual space. The different is revealed also by calculating the form of the change of the coordinates with the change of the base: vectors are transformed by the rule similar to Subsection 2.4 but the covectors are changed with the matrix base change, this is reflected in their names contravariant and covariant tensor respectively.

3.6 Determinant as the coefficient of expansion

If we consider the matrix form of (3.2) it is possible to give an interpretation analogous to the 2d case in Subsection 2.5, that is, it expresses the coordinates of the transformed vector as a result of the application of the affine transformation with the matrix determined by the system (3.2).

Then the determinant of the corresponding matrix obtains a very important geometric interpretation, namely, it gives the constant ratio of change of the volumes of the bodies under the affine transformation with this matrix.

To see this one may visualize the unit cube with one vertex at the origin, then the vectors of length, λ collinear to the edges of this cube, will be transformed into the vectors images according to formulas (3.2).

This cube of the volume λ^3 will be transformed into the parallelepiped constructed on these vector-images. Its volume, as has been discovered in subsec. 3.4.and 3.5 is equal to the triple product of these vectors, or to the determinant of the matrix. Comparing the volumes for arbitrary bodies (which can be approximated by the volumes the bodies constructed by cubes) one may state that that the coefficient of expansion is equal to the determinant of the matrix (taking into consideration its sign as well).

In the affine geometry it is impossible to express the volume of some body as some definite value, it is changed together with the coordinate system, therefore it is called “relative invariant”, because the relations between volumes of two bodies obey to the relations analogous to the

volumes in Euclidean geometry (generally the volume is defined by means of integrals, which is beyond of our scope of presentation).

In particular, the relative invariant V_D of the parallelepiped D in the n-dimensional affine space is expressed by the absolute value of the determinant constructed by coordinates of the vectors that form the parallelepiped. It can be shown that the ratio of volumes of two n-parallelepipeds constructed on vectors $\{\vec{a}_1,\ldots\vec{a}_n\}$ and $\{\vec{e}_1,\ldots\vec{e}_n\}$ is equal to the absolute value of the determinant of the matrix corresponded to the change of the second system to the first one. This has further application to skew-products of m-vectors and its geometric interpretation. This leads to more advanced interpretation of volume through the determinant of metric tensor in the case of Euclidean space.

4 CONCLUSIONS, FINAL REMARKS

We have illustrated how mathematical objects emerge from mathematical practices and how the nature of non-ostensive mathematical objects can be discovered in the preliminary stage of introduction to the linear algebra course.

Thus the students learn that the emergent objects exist and are independent of our will (like determinants and vector-product), moreover they lead to the introduction of some fundamental notions in linear algebra,

The students are able to discover the properties which characterize these objects and propose mathematical propositions that relate different objects.

We stress that The personal/institutional dialectic is essential in teaching processes, since it enables students to achieve the significant knowledge on the institutional level.

This is because we can use the language game that helps students to distinguish between personal objects and institutional objects, so that they participate in the constructing of mathematical knowledge, being convinced that this is an objective science.

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