PEDAGOGICAL GUIDANCE IN THE PROCESS OF CONSTRUCTION AND VISUALIZATION OF PRINCIPAL CONCEPTS IN THE B-LEARNING OF THE LINEAR ALGEBRA COURSE

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Abstract

In this article, we focus on the construction and comprehension of the concept of determinant which is one of the most difficult to grasp in the contexts of matrices as it is usually taught. Taking into account some theoretical elements from the Onto-Semiotic Approach and the significant learning in which the visualization takes the crucial role to achieve the comprehension of the concepts, we offer some instructional guides with a specific pedagogical objectives in order to provide an environment in b-learning modality where students could create personal significance of this concept and furthermore to be able construct the system of knowledge corresponding to the institutional significance in the Linear Algebra course.

Keywords: b-learning, determinants, OSA-Theory.

1 INTRODUCTION

In the process of construction of knowledge by the students in the B-learning modality the role of the professor is the most important although it depends in major part on the Theory of Learning adopted by the institution. At the Morelos State University, Mexico, much attention is paid to the formations of students as individuals, whereas the role of a professor is considered as accompanying in the construction of knowledge by the persons in the formation process. We accept the position of the author in [1] that the constructionist activity is the result of the interaction between the professor, expert in the field of knowledge, who provides the adequate influence through arrangement of the learning environment in the virtual space, and the students whose mental activity guided pedagogically allows to re-construct the content and to form the personal system of knowledge. Thus the pedagogical guidance through the instructional components of the course puts the tutor as the key element in the process of the construction of knowledge since the simple activity does not guarantee the significant construction of knowledge due to variety of factors, such as the lack of the cognitive recourses previous to realize the development and correct comprehension of the content [1] . On the other hand in order to produce the adequate instructional components the tutor needs some theoretical background as well as some tools to evaluate the efficacy of teachers' own practice. In the process of producing of the learning guides we follow the Onto-Semiotic Approach [2] for the analysis of our own practice.

The study of determinants in the theoretical aspects presents serious problems for the students of the Linear Algebra course in any of its modalities, present o virtual. Being aware of the crucial role which the visualization plays in learning demand to achieve the comprehension of a concept, we suggest some geometric interpretation which may be used to produce images that accompany the mathematical ideas employed in different context.

We offer some instructional guides with a specific pedagogical objectives in order to provide an environment where one could achieve the personal significance of this notion one of the most difficult to grasp in the contexts of matrices as it is usually taught (especially of the formalization of the definition for the nxn case) As well the matrix environment is also rather abstract for the students, although both concepts are of great importance in applications to economy and administration (as a technical tools) apart from its theoretical value in the mathematical proofs in many adjacent branches of mathematics. The Technology of b-learning provides the possibility of familiarizing with contents when a student is short of time or when they are interested more in the applications but need the true understanding of the mathematical nature of contents which underlay the implementation

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Finally, these instruction materials can be used for the autodidactic learners en which case the student usually have no previous knowledge, serving for the so-called long-life education.

1.1 Theoretical background

In order to produce the adequate instructional components the tutor needs some theoretical background as well as some tools to evaluate the efficacy of teachers' own practice. In the process of producing of the learning guides we follow the Onto-Semiotic Approach for the analysis of our own practice. In the Onto-Semiotic framework, teaching involves the participation of students in the community of practices sharing the institutional meaning, and learning is conceived as the students' appropriation of these meanings. In this regard we believe that this framework allows making an approach to the pedagogical guidance through the systems of practices [3].

The OSA is a theoretical framework used for the investigation in mathematical education which takes into account the triple aspect of mathematics as a socially shared problem-solving activity, a symbolic language and a logically organized conceptual system [2]. In OSA, the mathematical activity is modeled in terms of systems of operative and discursive practices. From these practices, the different types of related mathematical objects (language, arguments, concepts, propositions, procedures, and problems) emerge more complex objects, building cognitive or epistemic configurations among them.

The contextual factors to which the meanings of mathematical objects are relative and which attribute a functional nature to them the mathematical objects intervening in mathematical practices or emerging from them, depend on the language game in which they take part, and can be considered from the dual dimensions or facets: Personal-Institutional, Extensive–intensive, Ostensive–non-ostensive Expression–content [2]. The studies of these dualities, of the emergent objects of practices at first they will allow us offer some instructional guides with to specific pedagogical objectives.

1.2 Visualization

Being aware of the significant learning in which the visualization takes the crucial role to achieve the comprehension of the concepts, in the practices we call as visual the types of linguistic objects and artifacts involved in the visual perception. Although symbolic representations are visible inscriptions, we do not consider them as visual, but as analytical or sentential. In this sense, we point the role of visual practices and non-visual practices (analytical) in the activity and mathematical communication, in the formation of concepts [3], therefore we suggest some geometric interpretation which may be used to produce images that accompany the mathematical ideas employed in different context.

2 DETERMINANT 2X2 AS A CONCEPT EMERGING FROM COMPUTATIONAL PRACTICE

Institutional objects emerge from systems of practices shared within an institution, while personal objects emerge from specific practices from a person. ''Personal cognition'' is the result of individual thinking and activity when solving a given class of problems, while ''institutional cognition'' is the result of dialogue, agreement and regulation within the group of subjects belonging to a community of practices [2].

Our objective is to make evident a process of construction of fundamental concepts of the course Linear Algebra: determinant, matrix, range for the particular case of the linear equations 2x2. In the process of computational manipulations there emerge mathematical objects crucial to solve a particular problem: it is necessary to detect, name and use them for theoretical formulations and to justify its efficiency.

From the institutional point of view students should be able to distinguish between systems of linear equations (LES) which possess a single solution or an infinite set of solutions (the LES consistent). Applying a criterion constructed by the student, being formulated in terms of the coefficients of the system, to form and to calculate the determinant, the crucial concept to determine consistency (to be able to discriminate against the cases of existence the only or infinite solution.)

2.1 Operating Practice: Finding solutions of a linear 2x2 system

In order to solve the system of two equations on two unknowns.

$$\begin{cases} aX + bY = d \\ \alpha X + \beta Y = \delta \end{cases}$$

A student has to find out the values X=X_o Y=Y_o, such that both equations hold true.

Letting the students to apply computational techniques whatever they like, we suggest them to evaluate the complexity of their algorithms.

Usually they use the following practices taught at the school level:

1) To choose an unknown from one of the two equations and to express it, most frequently chosen action yields Y=(d-aX)/b
2) Substitute this expression in the other equation and obtain the equation on one variable $\alpha X + \beta(d - \alpha X)/b = \delta$, which, being simplify, gives $X(\alpha b - \beta a) = \delta b - \beta d$.

Some students prefer another practice: express the variable Y from both equations

$$Y = (d - aX)/b,\ Y = (\delta - \alpha X)/\beta,$$

And then to make the equality of right hand sides of both expressions, arriving at $X(\alpha/\beta - a/b) = \delta/\beta - d/b$, which is the same result as in the previous case.

Nevertheless, putting these ideas into concrete situations that give rise to such sort of system, they notice immediately that there "no sense" to express one unknown in terms of the other, because they denote the incomparable quantities.

Rather the computation should be made between the same symbols: thus they are led to the method of elimination of one of the unknowns from the other equations, when it is possible.

Thus making obvious manipulation they have,

$$\begin{cases} \alpha aX + \alpha bY = \alpha d \\ a\alpha X + a\beta Y = a\delta \end{cases}, \begin{cases} \alpha aX + \alpha bY = \alpha d \\ a\beta Y - \alpha bY = a\delta - \alpha d \end{cases}, \begin{cases} aX + bY = d \\ Y(a\beta - \alpha b) = a\delta - \alpha d \end{cases}$$

And analogously, by eliminating the unknown Y from the second equation, they obtain,

$$\begin{cases} aX + bY = d \\ X(a\beta - \alpha b) = -(b\delta - \beta d) \end{cases}$$

Actually in this process they realize two of three types of the so called "elementary transformations" of a system to another one which is equivalent to the initially given.

Thus they obtain the most simple system on two unknowns with an amazing feature, that the coefficient of each unknown have the same expression, moreover it is formed by means of the coefficients of the initial system according to a simple rule.

$$\begin{cases} (a\beta - \alpha b)X = (d\beta - \delta b) \\ (a\beta - \alpha b)Y = (a\delta - \alpha d) \end{cases}$$

2.2 Discursive practice: criterion of the consistency of a 2x2 system

At this very important stage the students are invited to explore the previous activities and to answer the following natural questions:

1) Is it true that the values which satisfy this system are those which are the solutions of the initial system?
2) What can we say about the solutions of the system which we have obtained?
3) Does it exist always? Is it unique? Does there exist any criterion?
4) Is it possible to suggest a nice way to express the solution by means of coefficients?

We would like to stress that the majority of students immediately express the variables dividing both parts by the coefficients, without paying attention whether the latter could be 0.

It is possible to have a unique solution only if $(a\beta - \alpha b) \neq 0$, $\begin{cases} X = (d\beta - \delta b)/(a\beta - \alpha b) \\ (Y = (a\delta - \alpha d)/(a\beta - \alpha b) \end{cases}$.

Thus this expression $\Delta = a\beta - \alpha b$ "determine" the existent of a solution and thus we can call it determinant, and denote $\Delta = \begin{vmatrix} ab \\ \alpha\beta \end{vmatrix} = a\beta - \alpha b$ (Δ is the capital Delta of Greek alphabet).

As a competition students try to express the nominators of the expression for the solutions, obtaining the Cramer formulas:

$$\begin{cases} X = \Delta_x / \Delta ,\ Y = \Delta_y / \Delta \end{cases} \quad \text{where } \Delta_x = \begin{vmatrix} db \\ \delta\beta \end{vmatrix} = d\beta - \delta b,$$

Furthermore their creativity permits to "invent" a "nice way" to construct the solution directly from the table of the coefficients of the initial system $\begin{pmatrix} ab \mid d \\ \alpha\beta \mid \delta \end{pmatrix}$

The necessity to have the positive answer to question 1) thus becoming obvious, as well students are fond of finding theoretical justification for the final conclusion.

$$\begin{cases} \Delta X = \Delta_x ,\ \Delta Y = \Delta_y \end{cases}$$

Most important of which is: it is not true that in case $\Delta = \begin{vmatrix} ab \\ \alpha\beta \end{vmatrix} = a\beta - \alpha b$ vanishes the solution does not exists.

3 PASSING TO CONSTRUCT AN ABSTRACT NOTION OF DETERMINANT

In this section we are focused in achieving the comprehension of the notion of determinant 3x3 and to conceive the idea of generalization to the cases 4x4 and nxn. Applying solutions of 2x2 system to 3x3 case students become aware of a process of the construction of fundamental concepts of the Linear Algebra course from the particular case of the linear equations 3x3 to further generalization.

An extensive object is used as a particular case of a more general class, which is an intensive object. The extensive/intensive duality is used to explain a basic feature of mathematical activity: the use of generic elements. This duality allows us to focus our attention on the dialectic between the particular and the general, which is a key issue in the construction and application of mathematical knowledge [2].

3.1 The notion of 3x3 determinan

Now the students a in position to pass to the case of the system on three unknowns

$$\begin{cases} ux + vy + wz = t \\ ax + by + cz = d \\ \alpha x + \beta y + \gamma z = \delta \end{cases}$$

Applying the results of previous case

May write directly the solutions of the last two equations $x = \frac{\begin{vmatrix} (d-cz)b \\ (\delta-\gamma z)\beta \end{vmatrix}}{\begin{vmatrix} ab \\ \alpha\beta \end{vmatrix}}$ $y = \frac{\begin{vmatrix} a(d-cz) \\ \alpha(\delta-\gamma z) \end{vmatrix}}{\begin{vmatrix} ab \\ \alpha\beta \end{vmatrix}}$, and

substitute the expression for unknown x and y to the first equation

$$u\left[(d-cz)\beta-(\delta-\gamma z)b\right]+v\left[a(\delta-\gamma z)-\alpha(d-cz)\right]+w\Delta z=t\Delta,$$

Simplifications give

$$(u\begin{vmatrix} bc \\ \beta\gamma \end{vmatrix} - v\begin{vmatrix} ac \\ \alpha\gamma \end{vmatrix} + w\begin{vmatrix} ab \\ \alpha\beta \end{vmatrix})z = (T)$$

As in previous exploration they introduce the “determinant”

$\Delta = \begin{vmatrix} uvw \\ abc \\ \alpha\beta\gamma \end{vmatrix}$, which is calculated by the rule suggested in the process of computational activities, namely

$\Delta = u\begin{vmatrix} bc \\ \beta\gamma \end{vmatrix} - v\begin{vmatrix} ac \\ \alpha\gamma \end{vmatrix} + w\begin{vmatrix} ab \\ \alpha\beta \end{vmatrix}$. Moreover after ingenious efforts of rearranging the right hand side term (T) they arrive to

$\Delta z = u\begin{vmatrix} b\delta \\ \beta d \end{vmatrix} - v\begin{vmatrix} ad \\ \alpha\delta \end{vmatrix} + t\begin{vmatrix} ab \\ \alpha\beta \end{vmatrix}$, which now can be written as $\Delta z = \Delta_z = \begin{vmatrix} uvt \\ abd \\ \alpha\beta\delta \end{vmatrix}$.

3.2 Passing to general case

The way to construct abstract concept as a generalizations of the particular cases is rather complicated. We outline the principle steps which allow the students to arrive to institutional level of corresponding representations.

For the case 4x4 and furthermore for nxn it is necessary to modify the notations representing the system (semiotic registers)

$$\sum_{i=1}^{i=n} a_{ki}x^i = b_k, k = 1,\dots,n.$$

Thus the definition of the determinant should be modified as well.

Thus the students are forced to examine the summands which gives the value of the determinant in 3x3 case, first.

$\Delta = u\begin{vmatrix} bc \\ \beta\gamma \end{vmatrix} - v\begin{vmatrix} ac \\ \alpha\gamma \end{vmatrix} + w\begin{vmatrix} ab \\ \alpha\beta \end{vmatrix}$ now will read $\Delta = a_{11}\begin{vmatrix} a_{22}a_{23} \\ a_{32}a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21}a_{23} \\ a_{31}a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21}a_{22} \\ a_{31}a_{32} \end{vmatrix}$

This formula appears as ready-made in the textbooks giving a way a calculation of determinants or adopted as a definition, so that the students should just remember it.

If one follows the rule for calculating the 2x2 determinants, then

$$\Delta_{3x3} = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{33} + a_{12}a_{21}a_{23} - a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31},$$

Analyzing the patron of how there appears each term students achieve the general formula for 3x3 determinant. Thus the students are prepared to understand the institutional level of presenting these themes in the textbooks.

4 VISUALISATION: NON-OSTENSIVE OBJECTS

The theoretical findings suggest that the visualization or its aspects like spatial imagination are indispensable for the development of the mathematical reasoning**.**

4.1 Theoretical foundations

Being aware of the significant learning in which the visualization takes the crucial role to achieve the comprehension of the concepts, in the practices we call as visual the types of linguistic objects and artifacts involved in the visual perception. Although symbolic representations are visible inscriptions, we do not consider them as visual, but as analytical or sentential. In this sense, we point the role of visual practices and non-visual practices (analytical) in the activity and mathematical communication, in the formation of concepts [3], therefore we suggest some geometric interpretation which may be used to produce images that accompany the mathematical ideas employed in different context.

Mathematical objects (both at personal or institutional levels) are, in general, non-perceptible. However, they are used in public practices through their associated ostensive (notations, symbols, graphs, etc.). The distinction between ostensive and non-ostensive is relative to the language game in which they take part. Ostensive objects can also be thought, imagined by a subject or be implicit in the mathematical discourse (for example, the multiplication sign in algebraic notation) [2]. On one hand, the mathematical object is immaterial, invisible, but it depends for his "existence" of the material, visible thing. This is one way to express the cognitive paradox of the mathematical learning that describes R. Duval.

"The visualization offers a method of seeing the invisible" This one "to "see" can be purely mental and then involves not-ostensive objects, or can be related to a physical representation to be a perceptible object.

This capacity to reveal unexpected truths is precisely the root in the usefulness of the algebraic formulations, for which its graphic-iconic character is the one that prevails. For example, the expression 2x - y + 1=0, is a straight line; the mere expression informs of the essential properties of the mathematical object**.** Usually the visual objects will take part in the mathematical practices together with other not visual objects (analytical or of another type). The visualization in mathematics does not diminish to seeing, but also it bears interpretation, action and relation**.**

The visualizing activity in mathematics includes the idea of the construction of a space of reference where the mathematical objects are placed. In this term, multiple cognitive processes are included such as interpretation, decoding, relation, linking, etc.

In case of the theoretical positions that will take place in this work, the visualization will be a skill to develop, a way to construct meanings as a direct actor of the process of thought, a function in the semiotic mediation between others, nevertheless, in all the postures that involve visualization as part of the learning, it is a necessary resource to learn the mathematics independently of the position of the researcher.

4.2 Geometric interpretation of determinants

Emerging in the context of a nxn system of linear equations, the determinant is the function of the coefficients of the system, which are arranged in the form of a table with n rows and n column, forming a “matrix” (“The Mother of the Determinant” as it was initially coined by J Sylvester).

But the column of real numbers can be given another interpretation, namely, as a column of coordinates of a vector in n-dimensional vector space.

In a setting a determinant permits to establish if the n-vector columns form a basis (i.e. linearly independent system), when $\Delta_{nxn} \neq 0$, then the sign of determinant determines the orientation of the base, and, in the Euclidean case, the absolute value of the $\det(a_{ki})$ gives the volume of the parallelepiped constructed on the vectors, whose coordinate form the matrix a_{ki} .

If a matrix is considered as the matrix of a linear transformation, then the unit cube constructed on the base orthonormal vectors will be transformed into parallelepiped, then the absolute value of determinant characterized the relation of both volumes.

Thus one may create a 3-dim picture of both geometrical figures which produce an image which illustrate the presence of a phenomenon of a determinant in this particular context.

Furthermore determinants have the very important geometric interpretation as the constant ratio of deformations of the volumes of all the bodies under linear transformation of affine space, which lead to visualization in the physics environment as expansion

5 FINAL REMARKS

Since determinants are related to matrices, in each case they admit different geometric interpretations. In the case of classifications of quadrics in 2 or 3-dim, the students are given specially constructed determinants whose signs in special combination permits to decide which curve o surface represents a second order relations between 2 o 3 unknowns, considered as coordinates of a point of real space.

Thus, pairs or triples of determinants, formed by coefficients of a general equation of second order, correspond to the variety of geometrical figures represented by the equation.

The explanation is related to the determinants of a matrix of a correspondent quadratic form based on the Criterion of Sylvester.

There are other appearances of determinants in variety of mathematical contexts, such as Gram determinant, Hessian etc., which will be a subject of further presentations.

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