The identification of the various partial meanings of a mathematical object and its articulation must therefore be a phase of the onto-semiotic analysis of mathematical activity. This analysis helps to formulate hypotheses about critical points in the interaction between the various agents in which there may be gaps of meaning or disparity of interpretations that require processes of negotiation of meanings and changes in the instruction process.

Batanero and Díaz (2007) apply the OSA theoretical notions to analyse the historical emergence of probability and its different current meanings (intuitive, classical, frequentist, propensity, logical, subjective and axiomatic). They, furthermore, describe mathematical activity as a chain of semiotic functions and use the idea of semiotic conflict to give an alternative explanation to some widespread probabilistic misconception.

Font and Contreras (2008) apply the notion of semiotic function and the OSA mathematical ontology to analyse the processes of generalization and particularization in the teaching and learning of mathematics. Using the analysis of the definition of derivative of a function in a high school textbook as a reflection context these authors address the following problems:

- The delimitation of particularization and generalization processes with respect to materialization and idealization processes;
- The elaboration of a typology of generalization processes;
- The role that the generic element plays in the relation particular–general;
- The relation of generalization processes with other mathematical processes.

Montiel, Wilhelmi, Vidakovic and Elstak (2009) apply OSA to analyse the mathematical notion of different coordinate systems, as well as some situations and university students' actions related to these coordinate systems in the context of multivariate calculus. The authors identify the objects emerging from the mathematical activity and make a first intent to describe an epistemic network for this activity. In other paper, Montiel, Wilhelmi, Vidakovic and Elstak (2012) approaches different coordinate systems through the process of change of basis, as developed in the context of linear algebra, as well as the similarity relationship between the matrices that represent the same linear transformation with respect to different bases.

4. Theories of meaning in mathematics education

The clarification of the notions of meaning and sense is a topic of interest for mathematical education and is approached from different perspectives. In this section, we synthetically describe three semiotic theories specifically oriented to mathematical knowledge: Frege's logical-semantic theory, Vergnaud's cognitive perspective and Steinbring's epistemological approach. Frege is a classic author who raises the distinction between sense and reference, which is a starting point for the Steinbring epistemological triangle, a model developed from a mathematics education explicit position. Vergnaud is representative of the theories of meaning from the constructivist psychological perspective. In these three theories, there is an interest in relating the question of the meaning of terms and expressions to the ontological problem on the nature of mathematical concepts, which is a central issue in the OSA. In Section 5, we analyse some concordances and complementarities between these semiotic theories and the OSA.

The concern for the meaning of mathematical terms and concepts leads directly to the inquiry into the nature of mathematical objects, to the ontological and epistemological reflection on the personal and cultural genesis and their mutual interdependence of mathematical knowledge. Reciprocally, behind any theory on the formation of concepts, or more generally, of any learning theory, there are ontological assumptions about the nature of concepts, and therefore, a more or less explicit theory about their meaning.

4.1. Sense and reference in Frege

Different triangular models have been proposed to analyse the relationships between symbols and meanings. One of them is introduced by Frege (1892) in the work *On meaning and reference*.

> It is natural, now, to think of there being connected with a sign (name, combination of words, letter), besides that to which the sign refers, which may be called the referent of the sign, also what I would like to call the sense of the sign, wherein the mode of presentation is contained (Frege, 1892, p. 210).

For example, let *a, b, c* be the segments joining the vertices of a triangle with the midpoints of the opposite sides. The point of intersection of *a* and *b* is also the intersection point of *b* and c (centroid). We, therefore, have different designations for the same point, and these names ("point of intersection of *a* and *b*" and "point of intersection of *b* and *c*") indicate at the same time the mode of presentation, and that is why the proposition contains effective knowledge. Both expressions have the same reference but different senses.

To each sign corresponds a given sense and to this sense, in turn, a specific reference, while a reference (to an object) neither is linked to only one sign, nor receives a single sense. The same sense has different expressions in different languages, and it may happen that an expression makes sense but has not a reference. For example, the expression "the series that converges more slowly" possess a meaning but has no reference, since for each convergent series, another series that converges more slowly can be found. Therefore, by grasping a sense, one is not sure there is a reference (Frege, 1892, p. 87).

Frege provides advice to distinguish between the reference and the sense of a sign, with regards to the representation associated with them, since the representation is something internal to subjects. If the reference of a sign is a perceptible object, then someone's representation is an image originating in the person's memories of sensorial impressions, and of internal and external activities. Even for the same person, not always, the same representation is linked with the same sense. Representation is subjective: the one's representation is not that from another.

> The sense of an expression was supposed to consist in the way in which we determined its reference: but now it appears that, often, there is no one favored way to determine the reference of an expression, but that different people may determine it in different ways, and even that what is taken at one time as an acceptable means of determining it may later be dropped as not agreeing with the others. If so, then what is objective about the employment of an expression, what is shared by all the speakers of the language, is after all its reference (Dummett, 1973, p. 102).

Initially, the theory of sense and reference was developed for proper names: "The referent of a proper name is the object itself which we designate by its means; the conception, which we thereby have, is wholly subjective; in between lies the sense, which is indeed no longer subjective like the conception, but is yet not the object itself" (Frege, 1892, p. 213).

Next, Frege expanded his theory of meaning and reference for declarative sentences, that is, statements that affirm a judgment to be true or false, and for common names or concepts. "Every declarative sentence concerned with the referents of its words is therefore to be regarded as a proper name, and its referent, if it exists, is either the true or the false" (Frege, 1892, p. 216).

Frege distinguishes between object and concept, which in logic is closely related to that of function, which Frege defines as: "A function of x was taken to be a mathematical expression containing x, a formula containing the letter x" (Frege, 1891, p. 138). For Frege: "a concept is a function whose value is always a truth-value" (Frege, 1891, p. 146); the values given to the argument of the function are the objects that fall under the concept. In logic, "we can designate as an extension the value-range of a function whose value for every argument is a truth-value" (Frege, 1891, p. 146).

In reference to the idea of object, Frege affirms: "An object is anything that is not a function, so that an expression for it does not contain any empty place” (Frege, 1891, p. 147). Value-range of functions are objects, whereas functions themselves are not. Likewise, extensions of concepts are also objects, although concepts themselves are not.

Frege's logical-semantic model distinguishes whether a sign refers to an object or to a concept, under a certain modality or meaning (sign, sense, reference). This is a first step in accepting that a concept admits a plurality of possible interpretations, uses or partial meanings. There is only an object/concept, but this object can be seen from different perspectives: for example, the centroid can be linked to the medians a, b of a triangle or to b and c.

Although Frege’s philosophy of mathematics is undoubtedly realistic - Platonist, when assuming that a mathematical object has its own independent existence, his theory of sense and reference of signs, words and expressions, opens a window to the relativism of the psychological and anthropological positions. One word designates or refers to an object or a concept, but it is always accompanied by a thought, one sense or specific way of seeing the object or concept in the context in which communication takes place. Such senses are considered in an intersubjective way, and consequently, the problem of identifying and characterizing the possible universe of senses attributable to the object can be raised.

4.2. Vergnaud’s conceptual triplet

A scientific challenge for Vergnaud (1982) is promoting the study of mathematics learning and teaching as a well-defined research field, not reducible to mathematics, psychology, linguistics, sociology or other sciences. This goal requires the analysis of the different mathematical contents, in their specificity, and the empirical study of their teaching and learning, in order to take into account both the long-term knowledge acquired by children and adolescents, and the short term change of conceptions when solving new situations. Consequently, Vergnaud produced the theory of conceptual fields in which he proposes a definition of concept useful to study the evolutionary development of mathematical knowledge. “Therefore, from a developmental point of view, a concept is altogether: a set of situations, a set of operational invariants (contained in schemes), and a set of linguistic and symbolic representations” (Vergnaud, 2009, p.94).

Vergnaud suggest that a concept cannot be reduced to its definition when we are interested in its learning and teaching (Vergnaud, 1990, p. 133). A concept acquires meaning for the child through situations and problems that should be solved. Vergnaud's study of a concept development and functioning, in the course of learning or during its use, lead him distinguish three planes or components, the triplet (S, I, G), as constituents of a concept C, where,

S: set of situations that give meaning to the concept (the reference).

I: set of invariants on which the operation of the schemes rests (the meaning)

G: set of linguistic and non-linguistic forms that allow symbolically represent the concept, its properties, situations and treatment procedures (the signifier).

There is neither general bijection between signifier and meaning, nor between invariant and situation. It is neither possible to reduce the meaning to the signifier, nor to the situation. The notion of meaning is understood as a relation of the subject to situations and the signifier. "More precisely, the schemes evoked by the individual subject in a situation or by a signifier are what constitute the subject’s meaning of this situation or signifier" (Vergnaud, 1990, p. 158). For example, the meaning of addition for a subject is the set of schemes that can be used to deal with the situations that face the subject which imply the idea of addition. It also includes the set of schemes involved when operating on the symbols (numerical, algebraic, graphic or linguistic) which represent addition.

Vergnaud (1982; 1990) goes one-step further than Frege in problematizing the mathematical concept when addressing the problem of learning and teaching: the concept itself is a complex, systemic entity formed by the interaction between three types of objects: representation systems, problem situations and operative invariants.

4.3. The epistemological triangle

Steinbring (1997; 2006) interprets Frege's triangle and that proposed by Ogden and Richards (1923), in adopting an epistemological perspective that helps to understand the interpretation, communication and construction of meanings processes that take place in the mathematics classroom.

His epistemological triangle includes three elements (Figure 4): the sign or symbol, the object or reference context, and the concept, which is understood as an ideal or abstract mathematical concept.

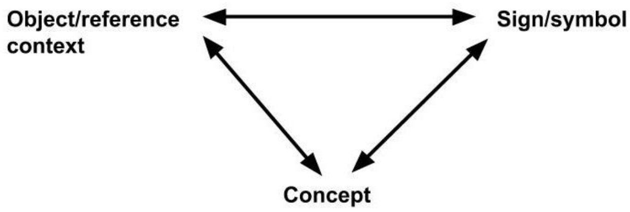

Figure 4. Epistemological triangle (Steinbring, 2006, p. 135)

Through the epistemological triangle, a semiotic (representational) mediation is modeled. "The links between the corners of the epistemological triangle are not defined explicitly and invariably, they rather form a mutually supported, balanced system. In the course of further developing knowledge, the interpretations of sign systems and their accompanying reference contexts will be modified" (Steinbring, 1997, p. 52).

Steinbring attributes two functions to mathematical signs:

> (1) A semiotic function: the role of the mathematical sign as "something which stands for something else".
>
> (2) An epistemological function: the role of the mathematical sign in the frame of the epistemological constitution of mathematical knowledge (Steinbring, 2006, p. 134).

To understand the Steinbring's semiotic-epistemological model of mathematical knowledge, the nature of the triangle vertices should be clarified. The author assumes that "The true mathematical object, that is the mathematical concept, may not be identified with its representations" (Steinbring, 2006, p. 137). Then, *what are for him the mathematical concepts? What are the objects/reference contexts*?

Steimbring's application of his epistemological triangle to the concept of probability (Figure 5) helps us to understand the features of this theoretical model of mathematical knowledge.

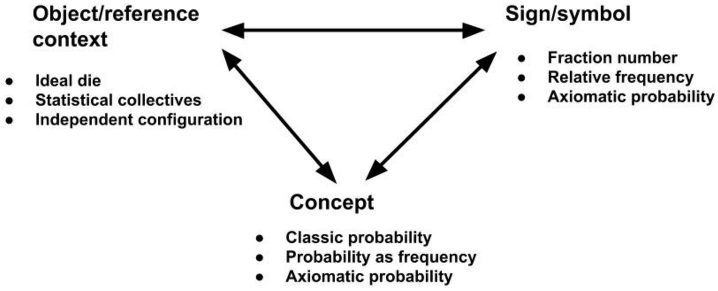

Figure 5. Application of the epistemological triangle to the concept of probability (Steinbring, 1997, p. 53)

In this figure, diverse expressions are included (fractional number, relative frequency, axioms) within the category of sign/symbol. Within the category of *objects /reference context* the model includes problem-situations where probability is applied, such as determining whether a given die is biased or not (ideal die), the statistical collectives to determine the probability, and the calculation of probabilities of independent event configurations. Within the category *concept,* it includes different meanings of the probability concept: classical, frequency and axiomatic probability meanings.

Thus, unless explicitly mentioned, it is assumed that the concept of probability has different meanings, depending on the contexts or situations-problems where it intervenes, and that such situations and meanings involve different systems of representation, although the reference to the axioms is very vague. Possibly, the author refers to the symbolic expression of the axioms, since the axioms themselves are not representations but properties of probability that link it with other mathematical objects, such as the union or intersection of events.

> The epistemological triangle is a model for making the invisible mathematical knowledge accessible with regard to its structural character, for describing its particularities and also for analyzing interactive processes of constructing mathematical knowledge – thus invisible relations that are embodied in exemplary contexts and activities (Steinbring, 2006, p. 144).

The Steinbring's epistemological triangle implicitly suggests, that the concept, reference signs/symbols and objects/reference setting, include a variety of general structures (various constructions of probability, natural number, etc.). The reciprocal relationships of conceptual structures with the representation systems and the different contexts and situations of use should be taken into account to organize and explain the generation of mathematical knowledge, that is, the concept epistemology. In this model, a systemic perspective is adopted, both for the structure of concepts and for the symbol systems and contexts. Frege attributes various senses to the mathematical concept, while for Steinbring these senses are reciprocally related to various symbolic systems and contexts.

5. Some concordances and complementarities between semiotic theories

The question of how the terms meaning and sense are used by various authors and disciplines is intriguingly and unclearly linked to the notion of object, and in the case of mathematics, to the nature of abstract objects. Therefore, semiotics is essentially linked with ontology, the various types of objects referred to by signs, and the various modalities in which objects can participate in communication and interpretation. The answers to the question of meaning of Frege, Vergnaud and Steinbring differ substantially in the nature of the objects referred to or represented by the signs, although the three models are triadic. In Frege assumes a platonic, transcendentalist position on the reference (the object referred to). The centroid, for example, is unique although it can be represented in different ways and each of them provides a different meaning. In a way, the models of Vergnaud and Steinbring respond similarly to the question of what represents, for example, the word 'number': it represents the (ideal, abstract) concept of number; but for the question of what number means, or what it is number the response is different: a heterogeneous system formed by three components (triplet): situations, invariants, representations (Vergnaud); the triplet sign, object, concept (Steinbring).

In the OSA, we find relevant differences in the response given to the question of meaning of a mathematical concept, in considering that these objects cannot be disentangled from the mathematical practices, and in assuming the anthropological perspective for mathematics, that is, conceiving mathematics as a human activity. In addition, objects and practices can adopt an institutional (social or shared practices), or personal (idiosyncratic practices of a subject) perspective and a systemic or unitary perspective.

When considering objects in a unitary way, their meaning would be one of the possible definitions (rules that intensively define the object). When viewing the objects systemically, the meaning is the system of operative and discursive practices in which said object intervenes in a critical manner, thus including one of the possible definitions, together with the situations, languages, properties and argumentations involved (partial meaning). The epistemological and didactic analysis of a mathematical object should also take into account the diversity of partial meanings of the object and their articulation in a *global meaning*, as can be seen in Batanero and Diaz (2007) for the probability or in Wilhelmi, Godino and Lacasta (2007) for the equality of real numbers.

Progress is made in the onto-semiotic framework towards an increasing refinement of the idea of mathematical concept by firstly connecting it with human activity, mediated by the linguistic and material artifacts put into play in the resolution of specific problem-situations. Secondly, each sense, or partial meaning, is linked to a specific rule (concept-definition) of the use of linguistic elements to solve a class of situations (contexts, phenomena) and other procedural, propositional and argumentative objects. Finally, the different partial meanings - senses - are organized in a global meaning formed by the network of senses and related objects.

The OSA also takes into account the functional or operational interpretations of meaning, that is, as the use of objects in the various practices. Thus, for example, numerical symbols do not only refer to the corresponding concepts, but they are also instruments for counting, numbering, ordering, etc.

6. Synthesis and implications

Along the paper, we have presented a synthesis of the theory of meaning that has been elaborated from the perspective of the OSA, which is supporting a new field of reflection on what could be called an *onto-semiotic* perspective, in which the study of signs should be linked to the analysis of the objects referred to by the signs. We have described the sources on which this onto-semiotics is based and the attempt to combine realistic and operational theories of meaning, since the problem is approached from the educational context, that is the context of construction, communication and learning of mathematical knowledge.

Although the problem of meaning interests to various disciplines (philosophy, linguistics, psychology, semiotics, etc.), the field of education, and, in particular, mathematics education, provides a rich perspective to address this problem. The OSA proposes not to separate the problem of signs and their interpretation from the ontological problem. This is understood in terms of inquiry about the nature and types of entities referred to by the signs, as well as the instrumental role played by them in the knowledge construction, communication and learning. The solution of the onto-semiotic problem, in turn, implies new ways of approaching the epistemological problem related to the origin and evolution of knowledge, which is undoubtedly essential to address the educational-instructional problem (Godino, et al., 2019).

In the OSA there are no objects without practices, which are made by people, and objects (concepts, propositions, representations, etc.) have a double "reality", personal (mental, cognitive), and institutional (cultural, shared). In this way, the OSA tries to articulate the cognitive with the epistemological problem (understood from an anthropological and therefore, historical-cultural perspective) in the mathematics learning.

In the curricular documents the terms meaning and sense are used informally, without really specifying what is meant by meaning or sense. In some cases, a cognitive mental vision of meaning is emphasized, while in others an epistemic/cultural vision is reinforced, when the design and analysis of mathematical instruction processes require articulating both perspectives. Consequently, we believe that the elaboration of a holistic theory of meaning that also takes into account the notions of sense and understanding is very useful for educational research and practice. This theory is applicable in the analysis of mathematical objects both at the macro level (such as when talking about developing in students the "numerical sense"), and at the micro level (in statements of the type, "students do not understand the meaning of the fraction concept, Pythagoras' theorem, the pie chart, etc.)". In the macroscopic analysis of any mathematical object it is necessary to take into account the various partial meanings and plan the learning of each one of them, as well as its progressive articulation. In addition, when solving problem-situations, in the processes of demonstration, representation, generalization, etc., teachers should consider the configuration of objects and meanings that students should know, interpret and understand.

Instructional design research or didactic engineering (Artigue, 1989; 2009) establish the phases of preliminary study, design, implementation and retrospective analysis in the teaching and learning processes. An epistemological analysis of the teaching contents is performed in the first stage, in order to have well-founded criteria to design the learning tasks in the second phase. The OSA theoretical tools, in particular the notion of partial meaning and its articulation to the global meaning, together with the categories of objects and processes involved in the practices (Figure 2) can support these phases of instructional design. As regards the implementation and retrospective analysis phases, the notion of didactic suitability offers a guideline to support the reflection on the teacher practice (Esqué & Breda, 2020; Morales-Maure, Durán-González, Pérez-Maya, & Bustamante, 2019).

The OSA to the meaning of mathematical objects has implications for teachers' education, since it highlights the complexity of knowledge and, therefore, the need to investigate pedagogical interventions, as well as making teachers be aware of this complexity.

> Teachers surely need mathematical content knowledge and pedagogical knowledge; and within the domain of pedagogical content knowledge, they also need epistemological knowledge so they are able to assess the epistemological constraints of mathematical knowledge in different social settings of teaching, learning, and communicating mathematics (Steinbring, 1998, p. 160).

In this sense, mathematics teacher should know the different meanings of mathematical objects, as well as the network of objects and processes involved in the mathematical practices, in order to be able to plan the teaching, manage the interactions in the classroom, understand the difficulties and assess the students' learning.

The notion of *didactic suitability* has been developed within the OSA framework (Godino et al, 2007; Breda, Font, & Pino-Fan, 2018) to provide criteria that mathematical instruction processes should fulfil to optimise learning, when taking into account the context constraints. With this aim, it is necessary to consider the knowledge provided by didactic research on the epistemic, cognitive, affective, interactional, mediational and ecological facets involved in the instructional processes. Moreover, the OSA pragmatic view of institutional and personal meanings provides an essential element for assessing the epistemic and cognitive suitability of mathematical instruction processes. On the one hand, to achieve high epistemic suitability the planned or implemented meanings should represent the global meaning of the mathematical object studied. On the other hand, an adequate level of cognitive suitability requires that the personal meanings constructed by the students agree with the institutional planned or implemented meanings (Pino-Fan, Font, Gordillo, Larios, & Breda, 2018), and the students are able to establish connections between the different meanings and objects involved.

This pragmatic approach to meaning can serve to analyse the representativeness of the meanings intended for a mathematical object given by the curriculum of a particular educational level in a given country (Pino-Fan, Parra-Urrea, & Castro, 2019; Burgos & Godino, 2020) or in textbooks (Burgos, Castillo, Beltrán-Pellicer, & Godino, 2020). In addition, introducing the students to a representative sample of the partial meanings of mathematical objects allows them to develop their problem solving competence, according to the different contexts in which these objects are used.

The notion of didactic suitability is being widely used as a tool to analyse the didactic sequences designed and implemented by teachers, in order to achieve an adequate teaching of mathematics (Breda, 2020; Morales & Font, 2019; Sousa, Silva Gusmão, Font, & Lando, 2020). It also is employed to organise training programmes focused on the reflection on teaching practice (Esqué & Breda, 2020; Morales-Maure, Durán-González, Pérez-Maya, & Bustamante, 2019). The didactic suitability construct facilitates the systematic reflection of teachers on the complexity of the mathematical objects they teach and on the factors involved in their study.

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References

Artigue, M. (1989). Ingenierie didactique. *Recherches en Didactique des Mathématiques, 9*(3), 281–308.

Artigue, M. (2009). Didactical design in mathematics education. In C. Winslow (Ed.),

Nordic Research in Mathematics Education. Proceedings from NORMA08 (pp. 7-16). Rotterdam: Sense Publishers.

Atkin, A. (2010). Peirce’s theory of sign. *Stanford Encyclopedia of Philosophy.* https://plato.stanford.edu/entries/peirce-semiotics/

Balacheff, N. (1990). Towards a "problématique" for research on mathematics teaching. *Journal for Research in Mathematics Education, 21*(4), 259-272.

Baker G. P. & Hacker P. M. S. (1985). *Wittgenstein. Rules, grammar and necessity. An analytical commentary on the Philosophical Investigations.* Basil Blackwell.

Batanero, C. & Díaz, C. (2007). Meaning and understanding of mathematics. The case of probability. In J.P Van Bendegen & K. François (Eds.), *Philosophical dimensions in mathematics education* (pp. 107-128). Springer.

Bloor, D. (1983). *Wittgenstein. A social theory of knowledge*. The Macmillan Press.

Blumer, H. (1969). *Symbolic interactionism: Perspective and method.* University of California Press.

Breda, A. (2020). Características del análisis didáctico realizado por profesores para justificar la mejora en la enseñanza de las matemáticas. *Bolema*, *34*(66), 69-88.

Breda, A., Font, V., & Pino-Fan, L. R. (2018). Criterios valorativos y normativos en la Didáctica de las Matemáticas: el caso del constructo idoneidad didáctica. *Bolema*, *32*(60), 255-278.

Brownell, W. A. (1947). The place of meaning in the teaching of arithmetic. *The Elementary School Journal, 47*(5), 256-265.

Bruner, J. (1990). *Acts of meaning*. Harvard University Press.

Burch, R. (2014). Charles Sanders Peirce. *Stanford Encyclopedia of Philosophy.* http://plato.stanford.edu/entries/peirce/

Burgos, M., Castillo, M. J., Beltrán-Pellicer, P., Giacomone, B., & Godino, J. D. (2020). Análisis didáctico de una lección sobre proporcionalidad en un libro de texto de primaria con herramientas del enfoque ontosemiótico, *Bolema*, *34*(66), 40-68.

Burgos, M. & Godino, J. D. (2020). Modelo ontosemiótico de referencia de la proporcionalidad. Implicaciones para la planificación curricular en primaria y secundaria. *Avances de Investigación en Educación Matemática, 18*, 1-20.

Campos, D. G. (2010). Peirce's philosophy of mathematical education: fostering reasoning abilities for mathematical inquiry. *Studies in Philosophy and Education, 29*(5), 421-439. DOI: 10.1007/s11217-010-9188-5.

Cobb, P. & Bauersfeld, H. (Eds.). (1995). *The emergence of mathematical meaning: Interaction in classroom cultures*. Lawrence Erlbaum Associates

Dummett, M. A. E. (1973). *Frege. Philosophy of Language.* Harper & Row, Publishers.

Dummett, M.A.E. (1996). *The seas of language*. Clarendon Press.

Eco, U. (1991). *Tratado de semiótica general*. Lumen, 1979.

Esqué, D. & Breda, A. (2021). Valoración y rediseño de una unidad sobre proporcionalidad utilizando la herramienta Idoneidad Didáctica. *Uniciencia, 35*(1), 38-54.

Font, V. & Contreras, A. (2008). The problem of the particular and its relation to the general in mathematics education. *Educational Studies in Mathematics, 69*, 33-52

Font, V., Godino, J. D., & Gallardo, J. (2013). The emergence of objects from mathematical practices. *Educational Studies in Mathematics, 82*, 97-12

Frege, G. (1891). Function and concept. In B. McGuinness (Ed.), *Collected papers on mathematics, logic, and philosophy* (pp. 137-156). Basil Blackwell, 1984.

Frege, G. (1892). Sense and reference. *The Philosophical Review, 57* (3) (May, 1948), 209-230.

Godino, J. D. & Batanero, C. (1994). Significado institucional y personal de los objetos matemáticos. *Recherches en Didactique des Mathématiques, 14*(3), 325-355.

Godino, J. D. & Batanero, C. (1998). Clarifying the meaning of mathematical objects as a priority area of research in mathematics education. In A. Sierpinska, & J. Kilpatrick (Eds.), *Mathematics education as a research domain: A search for identity* (pp. 177-195). Kluwer Academic Publishers.

Godino, J. D. Batanero, C., & Font, V. (2007). The onto-semiotic approach to research in mathematics education. *ZDM. The International Journal on Mathematics Education, 39*(1-2), 127-135.

Godino, J. D., Batanero, C., & Font, V. (2019). The onto-semiotic approach: implications for the prescriptive character of didactics. *For the Learning of Mathematics, 39*(1), 37- 42.

Godino, J. D., Font, V., Wilhelmi, M. R., & Lurduy, O. (2011). Why is the learning of elementary arithmetic concepts difficult? Semiotic tools for understanding the nature of mathematical objects. *Educational Studies in Mathematics, 77*(2), 247-265

Kutschera, F. von (1975). *Philosophy of language.* D. Reidel Publishing Comp.

Hjelmslev, L. (1943). *Prolegomena to a theory of language.* The University of Wisconsin Press, 1969.

Marrades, J. (2014). Sobre la noción de 'forma de vida' en Wittgenstein. *Agora, 33*(1), 139-152. http://dx.doi.org/10.15304/ag.33.1.1873

Montiel, M., Wilhelmi, M., Vidakovic, D. & Elstak, I. (2009). Using the onto-semiotic approach to identify and analyze mathematical meaning when transiting between different coordinate systems in a multivariate context. *Educational Studies in Mathematics, 72*(2), 139–160.

Montiel, M., Wilhelmi, M. R., Vidakovic, D. & Elstak, I. (2012) Vectors, change of basis and matrix representation: onto-semiotic approach in the analysis of creating meaning. *International Journal of Mathematical Education in Science and Technology, 43* (1), 11-32.

Morales-Maure, L., Durán-González, R., Pérez-Maya, C., & Bustamante, M. (2019). Hallazgos en la formación de profesores para la enseñanza de la matemática desde la idoneidad didáctica. Experiencia en cinco regiones educativas de Panamá. *Inclusiones, 6*(2), 142-162.

Morales, Y. & Font, V. (2019). Valoración realizada por una profesora de la idoneidad de su clase de matemáticas. *Educaçao e Pesquisa, 45*, 1-20.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* National Council of Teachers of Mathematics.

Ogden, C. K. & Richards, I.A. (1923). *El significado del significado.* Paidos, 1984.

Otte, M. (2006). Mathematical epistemology from a peircean semiotic point of view. *Educational Studies in Mathematics, 61*(1-2), 11-38

Peirce, C. S. (1931-58). *Collected Papers of Charles Sanders Peirce,* 8 vols. C. Hartshorne, P. Weiss, & A. W. Burks (Eds.). Harvard University Press.

Pimm, D. (1995). *Symbols and meanings in school mathematics.* Routledge.

Pino-Fan, L., Font, V., Gordillo, W., Larios, V., & Breda (2018). Analysis of the meanings of the antiderivative used by students of the first engineering courses. *International Journal of Science and Mathematics Education. 16*(6), 1091–1113.

Pino-Fan, L., Parra-Urrea, Y., & Castro, W. F. (2019). Significados de la función pretendidos por el currículo de matemáticas chileno. *Magis, Revista Internacional de Investigación en Educación, 11*(23), 201-220.

Presmeg, N. (2014). Semiotic in mathematics education. En S. Lerman (Ed.), *Encyclopedia of Mathematics Education*. (pp. 538-542). Springer.

Presmeg, N., Radford, L., Roth, W. M., & Kadunz, G. (2018). (Eds.). *Signs of signification. Semiotics in Mathematics Education Research.* Springer.

Radford, L. (2001). On the relevance of semiotics in mathematics education. Paper presented to the *Discussion Group on Semiotics and Mathematics Education at the 25th PME International Conference*, The Netherlands, University of Utrecht, July 12-17, 2001. Available from http://www.luisradford.ca/pub/91_On_the_relevance.pdf

Radford, L. (2006). The anthropology of meaning. *Educational Studies in Mathematics, 61*, 39-65.

Sáenz-Ludlow, A. & Kadunz, G. (2016). *Semiotics as a tool for learning mathematics. How to describe the construction, visualisation, and communication of mathematical concepts*. Sense Publishers.

Sierpinska, A. (1990). Some remarks on understanding in mathematics. *For the Learning of Mathematics, 10*(3), 24-36.

Sousa, J. R. de, Silva Gusmão, T.C.R., Font, V., & Lando, J. C. (2020). Task (re)design to enhance the didactic-mathematical knowledge of teachers. *Acta Scientiae, 22*(4), 98-120.

Speaks, J. (2014). Theories of meaning. *Stanford Encyclopedia of Philosophy.* https://plato.stanford.edu/entries/meaning/

Steinbring, H. (1997). Epistemological investigation of classroom interaction in elementary mathematics teaching. *Educational Studies in Mathematics 32*, 49-92.

Steinbring, H. (1998). Elements of epistemological knowledge for mathematics teachers. *Journal of Mathematics Teacher Education 1*, 157-189.

Steinbring, H. (2006). What makes a sign a mathematical sign? - An epistemological perspective on mathematical interaction. *Educational Studies in Mathematics, 61*, 133-162.

Ullmann, S. (1962). *Semántica. Introducción a la ciencia del significado.* Aguilar, 1978.

Vergnaud, G. (1982). Cognitive and developmental psychology and research in mathematics education: some theoretical and methodological issues. *For the Learning of Mathematics, 3*(2), 31-41.

Vergnaud, G. (1990). La théorie des champs conceptuels. *Recherches en Didactiques des Mathématiques, 10*(2,3), 133-170.

Vergnaud, G. (2009). The theory of conceptual fields. *Human Development, 52*, 83–94.

Wilhelmi, M. R., Godino, J. D., & Lacasta, E (2007). Configuraciones epistémicas asociadas a la noción de igualdad de números reales. *Recherches en Didactique des Mathematiques, 27*(1), 77-120.

Wittgenstein, L. (1953). *Philosophical investigations.* Basil Blackwell Ltd, 1958.

Wittgenstein, L. (1956). *Remarks on the foundations of mathematics.* The M.I.T. Press, 1967.